Pieces of Pi? Polyhedra, Orthoschemes and Dihedral Kaleidoscopes

Curtis Palmer M. Des., B. Sc.
Synergetic Design Inc.
Edmonton, Ab. Ca.
E-mail: clpalmer@shaw.ca

Abstract

Studying polyhedral forms is essential for mathematicians, architects, scientists, biologists, even artists, and for children it can be a lot of creative fun. This workshop will show that dihedral kaleidoscopes are useful tools for teaching mathematical concepts to a range of age groups. Workshop participants will experience creating a paper orthoscheme (also called: simplex, plug, quantum of shape, symmetry unit) and discover that polyhedra can be understood as products of kaleidoscopic reflections and rotations of such a simplex**,** see Coxeter [3]. The workshop will conclude with the collective creation of a paper polyhedra out of individualized, i.e. decorated simplexes. This transient sculpture will serve as visceral proof of the polyhedral consequences of symmetry operations.

Introduction. This workshop will provide educators with a hands on, experiential method for teaching mathematical concepts to students of varying age and receptivity. The use of cost effective classroom *manipulatives*: **dihedral kaleidoscopes** and paper models of **orthoschemes** will be demonstrated and encouraged.

Workshop Activity

Simplex really: A set of kaleidoscopes and orthoscheme *plugs* will be available for immediate viewing of the 27 polyhedra. Each workshop participant will be given a template of a plug, a **Symmetry Unit Net** (see Figure 3) of a truncated tetrahedron; to assemble and view in a kaleidoscope. Of these, twenty four of the individually decorated *quanta of shape* will be collected and conjoined to make the whole polyhedron. The floor will be open to discussion on classroom use of these manipulatives for varying ages and sophistication of students.

Pieces of Pi or Pieces of 8?

How 3 sets of 3 become 27. Each kaleidoscope's three mirrors are cut from a rational subdivision of a circle. Three kaleidoscopes are made with 3 sets of 3 mirrors intersecting at angles defined by the centre angles of the tetrahedron, octahedron and icosahedron, Figure 1. The tetrahedron's mirror set is cut from 4/8 of a circle, the octahedron's from 3/8 and the icosahedron's from 2/8 for a total of: 9/8 of a circle. (Is there a connection to be made with the whole tone of music? To Pirates?)

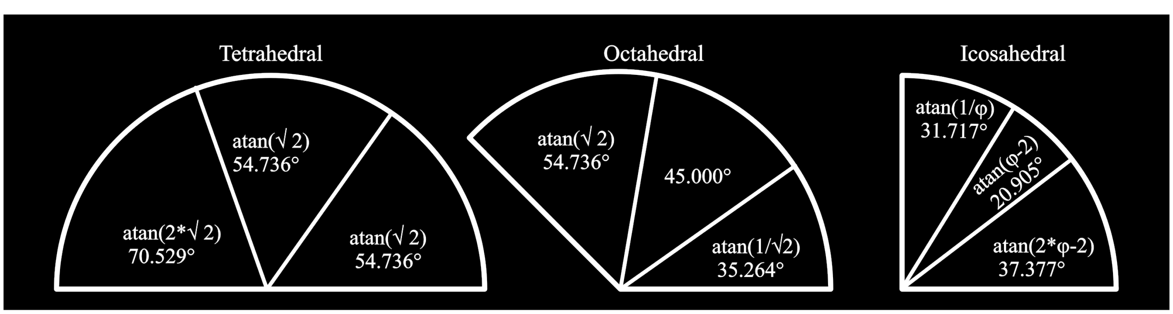

Figure 1: *The Centre Angles of Structure found in 9/8 of a circle*

The patterns of Figure 1 when cut, folded and joined along the free edge form a set of irregular tetrahedra with great circle arcs delimiting a void. With the mirror surfaces on the inside of the void, polyhedra can be modeled by introducing an orthoscheme *plug* between the reflecting surfaces. The *orthoscheme* is that smallest slice of the polyhedron, a *quantum of shape* that when copied with the symmetry operations: reflection and rotation, becomes the whole *solid*. In these kaleidoscopes one sees an *image* of the front side of a polyhedron, see Figure 5.

Polyhedral Families. With these 3 mirror assemblies we can show kinship by kaleidoscope symmetry between 27 classical geometric forms: 5 platonic, 11 archimedean and 11 catalan, Figure 2, included in Palmer [7], an *e-handout* (pdf) for workshop participants.

Tetrahedral Family (2of 3)			
Tetrahedron	*F4 V4 E6*	Tetrahedron	*F4 V4 E6*
Octahedral Family (2 of 12)			
Octahedron	*F8 V6 E12*	Hexahedron or Cube	*F6 V8 E12*
Icosahedral Family (2 of 12)			
Icosahedron	*F20 V12 E30*	Dodecahedron	*F12 V20 E30*

Figure 2: Kaleidoscopic kinship: rows = duals with topological features: *F*aces, *V*ertices and *E*dges

3D to 2D: Derivation of an Orthoscheme

The derivation of a tetrahedral orthoscheme is framed in Figure 3; instructions on reading the figure follow. The text that teachers use with different students to teach this material will of necessity vary, to accommodate individual levels. The multiple synonyms of *orthoscheme* used in this paper underline the need to find appropriate language for the target audience. During the workshop, participants will have hands on experience assembling and decorating one *orthoscheme,* a truncated tetrahedron; viewing their simplexes in a kaleidoscope; and collectively constructing the complete polyhedron from individualized orthoschemes. For the complete set of 27 orthoscheme derivations, see the *e-handout*.

Reading Figure 3

Verso. The left page of Figure 3 names the polyhedron from which we derive its orthoscheme. The chosen **Profile,** one of many possible projections, matches the smallest hole in a plane that this polyhedron could pass through (inspired by the game *TetraToss*). The shaded area in the profile represents the part of an orthoscheme that contributes to the polyhedral face(s). The **Net** is one of many possible layouts for flattening the polyhedra's surfaces onto a plane. The shaded area referencing the orthoscheme is repeated in the net diagram. Cut, fold and tape the Net to model the polyhedron. The **Symmetry Unit Net** graphs the 'folding to the plane' of the orthoscheme from within a polyhedral face (blue). This *quantum of shape* links the shaded area (polyhedral surface planes) and the three axial planes connecting these surfaces to the polyhedron's centre, its origin. Cut, fold and tape the *Symmetry Unit Net* to produce the orthoscheme *plug*. Build enough *plugs* (24 tetrahedral, 48 octahedral, 120 icosahedral) and you can paste them together to model a polyhedron. Remember the *enantiomorphs!*

Recto: Orthoscheme Analysis. Unique to each polyhedron, the orthoschemes are reduced to various components described on the right side of Figure 3. A list of angular and linear values are echoed by simple graphics and referenced to an **Index** of unique trigonometric functions (Figure 4). In the graphics

I have maintained a standard use of colour and line weight to assist in reading the relationships. These colours are lost in the Bridges B&W proceedings. The diagrams at a simple level are for appreciating the visual proportions of: line, angle, area, and shape of the component parts of an orthoscheme.

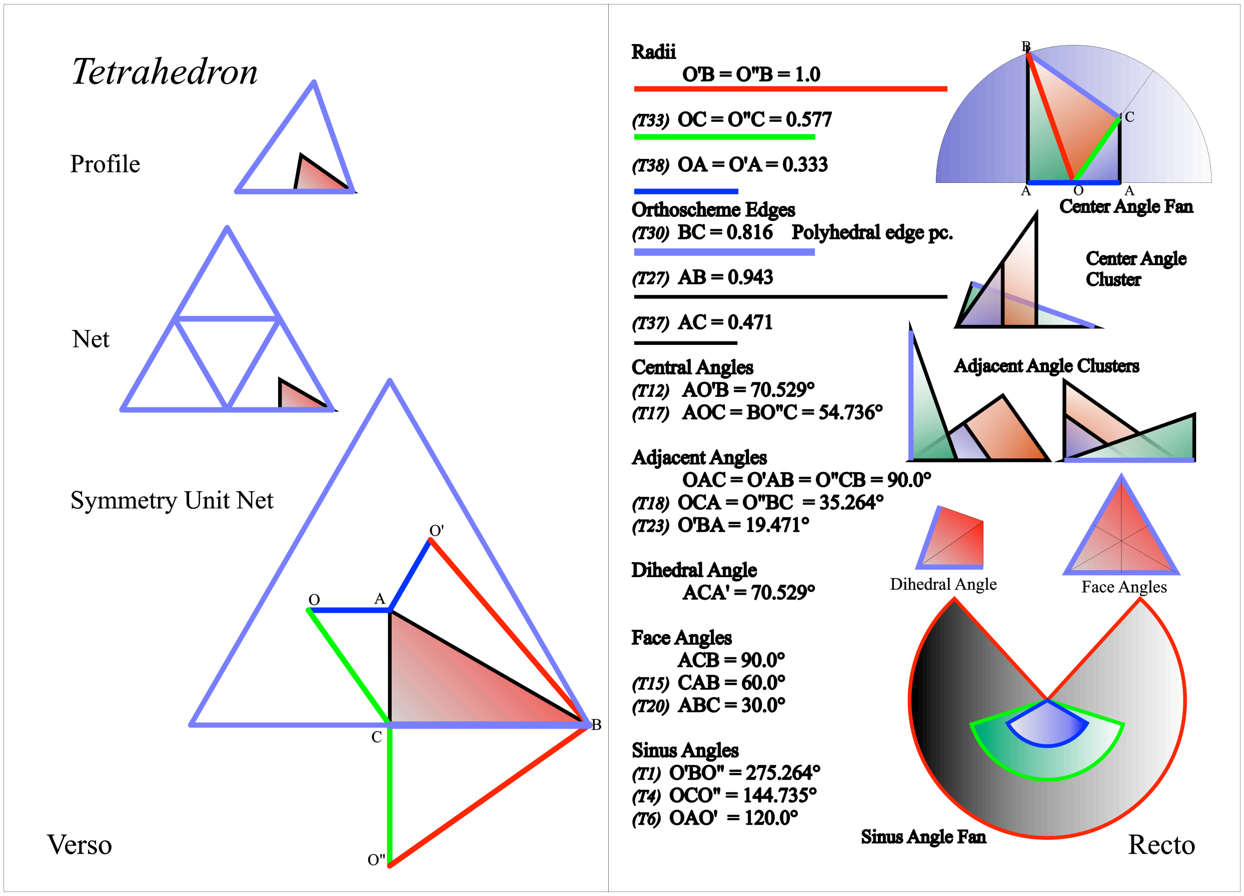

Figure 3: *Tetrahedral orthoscheme derivation and analysis*

The **Radii**, in sets of 3 to 5, identify the: circumsphere - red; intersphere - green; insphere - blue; and intermediate spheres less rigorously defined - black. The polyhedral **Edges** - thick blue. The **Centre Angle Fan** (top right) maps the orthoscheme's axial plane triangles onto the cutting planes of the mirror assembly folded flat; the polyhedral faces (the shaded areas, Verso) degenerate to edge(s) perpendicular to the page. The radii become the cuts that produce the **Sinuses** required in constructing the orthoscheme from a flat piece of paper. These axial plane triangles are reproduced in various *clusters* to visually practice with their proportions. Similarly **Dihedral Angles**, **Face Angles** and **Sinus Angles** are graphed to accompany the tabular data; again, for visual practice.

F#	Function	Radians	Degrees
T12	acos(1/3)	1.231	70.529
O53	$\text{atan}(\sqrt{2})^{-1}$	0.615	35.264
I63	atan(1/φ) where $\varphi=(1+\sqrt{5})/2$	0.554	31.717

Figure 4: *Numbers Index - excerpted - select values from Figure 1*

The Indexes: Figure 4 is an excerpt of the full table included in the *e-handout*. These tables include the 39 tetrahedral, 118 octahedral, the 144 icosahedral numbers arising from the orthoscheme analyses. The functions can be used to calculate these values to any desired accuracy and they should serve students interested in the numerical basis of polyhedral symmetries with repeated encounters of *transcendental numbers,* e.g. the golden ratio.

Notes on Constructing Dihedral Kaleidoscopes

Material Considerations: A practical set of classroom kaleidoscopes needs to be: cheap, safe, and sized to fit both young hands and orthoschemes constructed from a single sheet of bristol board. First surface mirrors are optically best although glass can shatter and draw blood. I use thin aluminum sheet with a highly reflective surface purchased from luminaire manufacturers. It is lightweight and can be accurately cut. It can be backed by a rigid foam, to hide sharp edges and taped together, to rapidly assemble the mirror pieces into a functioning dihedral kaleidoscope.

Conclusion

Classroom Usage: I have presented this material to elementary & junior high students and an undergraduate industrial design class. From this exposure to students' needs and abilities, I offer these conjectures on which students might benefit from *Pieces of Pi* in the classroom. For the students of:

- Craft: a 3D quilting bee, promoting hand eye coordination and collective creation;
- Architecture: an introduction to space frames and space fillers, see Critchlow [4];
- Rock: an introduction to crystallography, chemistry and the whole tone, see Smith [9], Ashton [2]
- Life: a look at viral protein shells and tensegrity in organelles, see Ingber [5];
- Industrial design: enantiomorphism (just because!), see Kappraff [6];
- Number theory: trigonometry, topology, and symmetry, see Ash & Gross [1];
- Geodesy: an introduction to great circles, Dymaxion Maps and the geoscope, see Palmer [8].

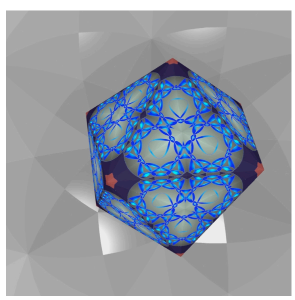

Figure 5: *A virtual icosahedron: Rhinoceros render of 4 surfaces by the author, 2005*

References - Further Reading

[1] A. Ash and R. Gross, *Fearless Symmetry*, Princeton University Press, 2006.
[2] A. Ashton, *Harmonograph*, Walker and Company, 2003.
[3] H.S.M. Coxeter, *Regular Polytopes,* Dover Publications, pp. 137. 1973.
[4] K. Critchlow, *Order in Space,* London Thames. 1976.
[5] D.E. Ingber, *The Architecture of Life,* Scientific American, pp. 48-57. Jan. 1998.
[6] J. Kappraff, *Connections*, McGraw Hill, 1991.
[7] C.L. Palmer, *Dihedral Kaleidoscopes,*. Self published 1996.
[8] C.L. Palmer, *Omniopticon: Design Alternatives for a Spherical Projection System*, U. of A., 1994.
[9] C.S. Smith, *A Search for Structure*, MIT Press, 1982.