The Dynamics of Grid Square Dances

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Abstract

Grid square dances share features of both American square dances and contra dances. A number of four-couple "squares" are initially arranged in a grid on a large dance floor. Each dance is short and performed multiple times; an iteration of the dance ends with some couples "progressing" into a neighboring square. Grid squares are relatively new dances, and there is not an established manner of progressing from square to square. Rather, choreographers such as Bob Isaacs, Rick Mohr, Kathy Anderson, and Carol Ormand have experimented with different patterns of movement within and between squares. We will discuss the aesthetics of grid square progressions and use techniques from the study of discrete-time dynamical systems to analyze particular dances.

1. Introduction

Traditional American square dances are performed by four couples initially arranged in a square, with each couple (two people) forming a side. During the course of the dance, a "caller" directs the eight dancers in the square to interact in intricate "figures," either moving as individuals or as couples. A dancer may visit several places within the square, but by the end of the dance the caller returns everyone to his or her starting position and partner. A skilled caller is more than just a director; he or she choreographs dances and may even improvise sequences of figures ("calls") on the fly. Contra dances are similar to squares in that dancers start the dance arranged by couples and perform figures as directed by a caller. However, instead of belonging to one four-couple square, contra dance couples are arranged in long lines containing an indeterminate number of couples and extending along the dance hall away from the caller and band. These long lines are then broken into small two-couple squares. Dancers interact with the three other people in their small square during one time through the dance. Depending on their starting position, couples "progress" in either direction along the line after one iteration of the dance; the small squares re-form and the dance continues. Unlike square dances, contra dances are short—each dance takes about thirty seconds—but repeated many times, so that each couple gets a chance to move to all the different positions in the line. Typically, the dance repeats at least as many times as the number of couples in line; this number may be fifteen or more.

Grid squares, the subject of this paper, share features of both square dances and contra dances. The word "grid" refers to the fact that a number of four-couple squares are initially arranged in a matrix on a large dance floor. Like square dances, each couple interacts with three others during one course of the dance. However, unlike traditional square dances—and like contra dances—the dance is short and performed multiple times; after each iteration, some couples "progress" into a neighboring square. (It is essential that squares maintain their location in the grid, so each square has well-defined neighbors.) Grid squares are relatively new dances, and there is not an established manner of progressing from square to square. Rather, callers such as Bob Isaacs, Rick Mohr, Kathy Anderson, and Carol Ormand have experimented with different patterns of movement within and between squares. We will discuss the aesthetics of grid square progressions and use mathematics to analyze some possible patterns. However, we will first turn to the simpler case of contra dancing, which inspired callers to choreograph grid squares.

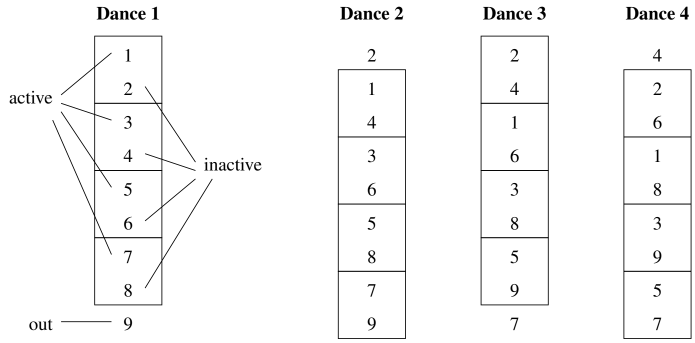

Figure 1: *Contra dance with nine couples.*

2. Contra dances

Any progressive couple dance, like a contra dance or grid square, can be described mathematically by *actions* (dance figures) that rearrange members of a *set* (people). If we were to represent the entire dance faithfully, the set would consist of individual dancers and the actions would be the various figures of the dance. Harkleroad [2] and Mui [4] take this approach in their analyses of contra dances, while Buchler and Rogers [1] do the same for square dances (von Renesse and Ecke [5] provide an interesting contrast; they use mathematical notation to represent salsa moves and examine possible sequences of moves). In this investigation, we only care how couples progress at the end of each iteration of the dance, so we'll say that our set consists of couples and our actions are progressions. (We need only keep track each couple as a unit because each person progresses together with his or her partner.) We want to track the position of the couples on the dance floor at the beginning of each iteration of the dance; this is a *state* of the system. Since we are only recording a couple's position at discrete time intervals (that is, at the beginning of each time through the dance), this model is a discrete-time dynamical system. One feature of the dance that we would like to track is how an individual couple moves through the dance floor; this is the *orbit* of that particular couple. In the course of each repeat of the dance, a couple typically interacts only with the couples in its four-person square (called a *minor set*); the sequence of couples that a given couple meets are its *neighbors*.

Let's see how this terminology describes a standard contra dance. We give each couple a number indicating its original position in line, starting at the "top" (the end of the line closest to the caller and band). Although the relative position of the couples changes throughout the dance, these numbers are fixed. Before the music starts, the dancers "take hands four," meaning that they hold hands in minor sets of four people. Figure 1 (left) represents a contra line of nine couples arranged in four minor sets. The couples at the tops of the minor sets are "actives," and the other couples are "inactives." (In traditional contra dances, the active couple has a more interesting role in the dance; the trend in modern dances is to give each couple an equally satisfying part.) If there is an odd number of couples, the bottom couple (here, couple 9) waits out the first dance but maintains its place in line. During the first time through the dance, the members of each minor set dance together. By the end of the first dance, active couples have traded places with the inactives in their minor set and formed new minor sets with the next inactive couples below them. If there is no one available to dance with—this can only occur at the top or bottom of the dance line—the couple waits out one time through the dance, then re-enters, taking the opposite role: actives become inactives and vice versa. For example, in the second dance, the original couple 2 is out at the top, while couple 9 becomes the inactive couple of the fourth minor set. The positions of couples 1 through 9 in the first four times through the dance are shown in Figure 1.

If we call the contra dance progression P, then we could say that $P(1) = 2$, $P(2) = 1$, etc. The *orbit* of couple 2 is its sequence of positions.

$$\text{Orbit}(2) = P^0(2), P^1(2), P^2(2), \ldots = 2, 1, 1, 2, 3, 4, 5, 6, 7, 8, 9, 9, 8, 7, \ldots$$

One thing that's fun about contra dancing is that each couple moves around a lot. The orbit shows that, in a nine-couple line, eighteen times through the dance are required for each couple to dance in every possible position and return to its starting point. Another thing that's fun is that you get to dance with a lot of different neighbors. For example, couple 2 dances first with couple 1, then is out at the top (indicated by the empty set $\emptyset$), then dances with couple 4:

$$\text{Neighbors}(2) = 1, \emptyset, 4, 6, 8, 9, 7, 5, 3, 1, \emptyset, 4, 6, 8, \ldots$$

After nine iterations, each possible pair of couples has met in the same minor set at some point.

"Double progression" is a variation on the standard contra dance progression. Each active couple meets *two* inactive couples in the course of one iteration of the dance. Many double progression dances work the same way as single progression dances, except that the length of one iteration is half that of a standard dance. However, in some double progression dances, the active couple dances with one inactive couple for the entire iteration, changes places with them, then quickly passes by the next inactives to start the dance with the second inactive couple below. We can track the orbit and neighbors of couple 2 in this situation, for a nine-couple set:

$$\begin{aligned}\text{Orbit}(2) &= P^0(2), P^2(2), P^4(2), \ldots = 2, 1, 3, 5, 7, 9, 8, 6, 4, 2, \ldots \\ \text{Neighbors}(2) &= 1, 4, 8, 7, 3, \emptyset, 6, 9, 5, 1, \ldots\end{aligned}$$

With a nine couple set, each couple has met every other couple, danced in every position, and returned home after only nine iterations of the dance. In this example, double progression is a more "efficient" way of moving couples on the floor, in that it takes less time for couples to dance in all the positions, while still meeting all the other couples in the minimum amount of time. However, the benefits of double progression depend on having an odd number of couples. Let's see what happens with eight couples. After eight times through the dance, every couple has danced every possible position and returned home—but notice the sequence of neighbors of couple 2:

$$\begin{aligned}\text{Orbit}(2) &= P^0(2), P^2(2), P^4(2), \ldots = 2, 1, 3, 5, 7, 8, 6, 4, 2, \ldots \\ \text{Neighbors}(2) &= 1, 4, 8, 5, 1, 4, 8, 5, 1, \ldots\end{aligned}$$

This situation is not limited to couple 2; all couples only meet four of the seven available neighbors. In fact, we could partition the set into two "communities": couples 1, 4, 5, 8 and couples 2, 3, 6, 7. Each couple's neighbors are the members of the other community. Callers don't typically control the number of couples in a contra line, so there is no good way to avoid this situation, other than not calling double progression dances of this type.

This discussion has introduced two important qualities of good choreography: *movement*—occupying a large number of possible positions on the dance floor—and *mixing*—dancing with a good number of other couples. We can measure movement by the number of different positions that a couple's orbit contains and mixing by the number of different couples that any given couple encounters during the dance. Double progression contra dances with an odd number of couples are optimal in both respects: each couple moves to all possible positions on the dance floor and meets all the other couples as efficiently as possible. However, double progression dances with an even number of couples do not mix the dancers thoroughly. We have seen that, in addition to the type of progression, the number and arrangement of couples sometimes determines how well movement and mixing are achieved.

3. Grid squares

In a grid square dance, four-couple square sets are arranged in a grid, which can be either rectangular or rectangular with a corner subtracted. These squares play the role of minor sets in contra dances. Figure 2 (left) shows one possible arrangement of eight squares into a "3+3+2" grid. Viewed from above, *head couples* are in the top and bottom positions of each square and *side couples* are in the left and right positions. In the top left square, couples 1 and 3 are heads and 2 and 4 are sides. During one time through the dance, couples change places with other couples in their square—their neighbors—before progressing to a square that is adjacent in the grid. The challenge is to write a grid square dance that achieves both movement and mixing without being overly difficult, either for the dancers or the caller. Ideally, a grid square algorithm should be robust in that the amount of mixing and movement is not too sensitive to the number or arrangement of square sets.

3.1. Progressions and permutations. In one iteration of a standard contra dance, the two couples in a minor set change places, then form a new set with the nearest couple in an adjacent square, if one exists. Grid squares generalize this sequence. Each dance consists of a *permutation*, which rearranges couples within a square, followed by a *progression*, in which some or all couples exchange positions with couples in adjacent squares. Whatever permutation is chosen acts in the same way on all the squares; in contrast, progressions act differently depending on whether a couple is on the edge or in the middle of the grid. Unlike couples in contra dances, couples in grid squares are never "out," because the number and position of squares in the grid does not change. The progressions that have been used for grid squares include *heads swap* (H), *sides swap* (S), *both swap* (HS), *head diagonals swap* (D), and *side diagonals swap* (D'). These are explained in Figures 2 and 3.

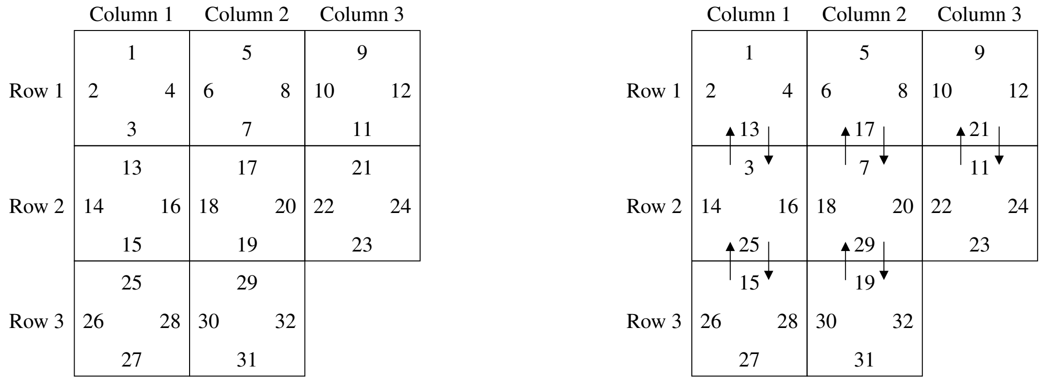

Figure 2: *(Left) 3+3+2 grid square. (Right) The result of the heads swap progression is that head couples have changed places with head couples in adjacent squares, if available. The sides swap progression is similar, but side couples change places with side couples. For example, couples 4 and 6 change places in a sides swap move. "Both swap" means both heads and sides swap.*

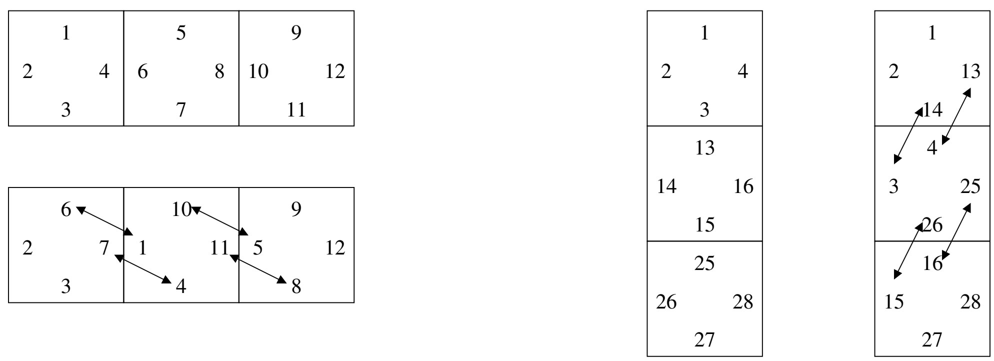

Figure 3: *The side diagonals swap (left) and head diagonals swap (right) progressions.*

Since progressions involve movement to an adjacent square, permutations are essential to move dancers away from the edges of the grid where there are no adjacent squares (you might have noticed that poor couple 2 doesn't move in any of the progressions we discussed). Although any of the 24 permutations of four couples is theoretically possible, only a few are used in actual dances: rotation, inversion, and their combinations. Rotation (R) moves all couples one place to their left within their square (so the couple in position 1 moves to position 4, the couple in position 4 moves to position 3, etc.) and inversion (I) exchanges the couple in position 1 with the couple in position 4 and exchanges the couple in position 2 with the couple in position 3. Rotation and inversion can be combined in various ways; they generate the (noncommutative) dihedral group D_4, which is the symmetry group of a square. For example, three rotations (R^3) is equivalent to moving one place to the right, while the result of inversion followed by two rotations (R^2I) is to exchange the couple in position 1 with the couple in position 2 and exchange the other two couples.

4. Dances

Table 1 summarizes the available moves and their mathematical notation. We'll use the mathematical convention of composing actions right to left, so that "do R then I" is written "IR."

"Warp and Weft" (Rick Mohr [3]). The dance alternates between two progressions, DI and $D'IR^2$. In the first of these options, couples move along the columns of the grid, while in the second, they move along the rows of the grid. In either case, couples either progress into the same role in an adjacent square or "bounce" at the edges of the grid and take on a different role. The orbits for "Warp and Weft" are quite complicated. Figure 4 (left) shows the orbit of the first couple, starting and ending at the black dot. In the course of forty-eight iterations of the dance, each couple visits every possible position on the dance floor before returning home. (Of course, it is unlikely that the dance would actually run forty-eight times!) The fact that the amount of movement for this particular grid is maximal is a consequence of the dimensions of the grid (3 and 4) being relatively prime. Rectangular grids with dimensions that are

Permutations

R	Rotation	Each couple moves one position to their left within their square
R^2		Each couple exchanges places with the couple directly across the square
R^3		Each couples moves one position to their right within their square
I	Inversion	Couples in positions 1 and 4 trade places; couples in positions 2 and 3 trade places
IR		Side couples (2 and 4) trade places
IR^2		Couples in positions 1 and 2 trade places; couples in positions 3 and 4 trade places
IR^3		Head couples (1 and 3) trade places

Progressions

H	Heads swap	Head couples swap into adjacent squares, if available
S	Sides swap	Side couples swap into adjacent squares, if available
HS	Both swap	Heads swap with adjacent heads and sides swap with adjacent sides, if available
D	Head diagonals swap	Head diagonals (positions 1 and 2) swap with nearest side diagonals (positions 3 and 4) in available adjacent squares in the same column of the grid
D'	Side diagonals swap	Side diagonals (positions 1 and 4) swap with nearest side diagonals (positions 2 and 3) in available adjacent squares in the same row of the grid

Table 1: *Mathematical notation for grid square permutations and progressions.*

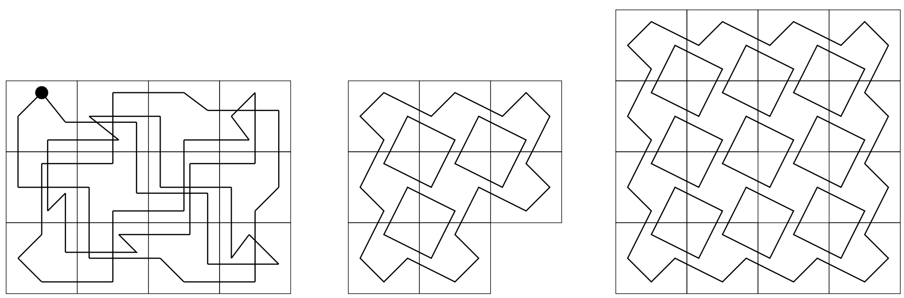

Figure 4: *(Left) Orbit of couple 1 for "Warp and Weft" generated by repeatedly applying DI and $D'IR^2$. (Center and right) Orbits generated for "Can of Worms" by repeatedly applying HSR on 4×4 and $3+3+2$ grids.*

not relatively prime yield suboptimal orbits. However, even for a 3×4 grid, the amount of mixing is smaller than one might expect. For example, in the eighth, twenty-fifth, and thirty-second iteration of the dance, the original squares re-form, albeit in different positions on the grid. (This is not necessarily bad: it's fun to meet one's original neighbors in unexpected places!)

"Can of Worms," "Swing Tunnels," "Getaway Hey," "Go Play in Traffic," "Balance the Grid," "Moving Target" (Bob Isaacs). These dances normally involve a clockwise rotation R followed by the progression HS, meaning that both heads and sides swap into adjacent squares, if available. If this pattern continues, two types of orbits appear, as shown in Figure 4 (center and right): several small counterclockwise period-four orbits and one large orbit that goes clockwise around the outside of the grid. This situation persists no matter the size of the grid. The amount of mixing is not optimal; for example, couples 1, 2, and 3 travel in the same orbit and are frequently in the same square, while interior couples meet each other every four times through the dance.

In order to increase the amount of mixing and movement, Bob occasionally rotates the square three places (R^3) instead of one place (R) before the progression HS. One question we wondered was "what is the optimal number of times to wait before varying the rotation?" Figure 5 shows that using R^3 instead of R every fourth time through the dance is an improvement. No couple has an orbit smaller than eight positions. However, this strategy introduces a "corner effect": four couples get trapped in the corner of the grid for three iterations of the dance.

Figure 6 shows some possibilities that are clearly problematic. Repeated application of the sequence HSR, HSR, HSR^3, HSR^3 (left) generates plenty of large orbits; however, many of the couples who start along the edges of the grid are trapped in small orbits of only four positions. On the right, the sequence HSR, HSR^3 produces orbits that traverse the grid diagonally (half the couples move along diagonals perpendicular to the ones shown). Four

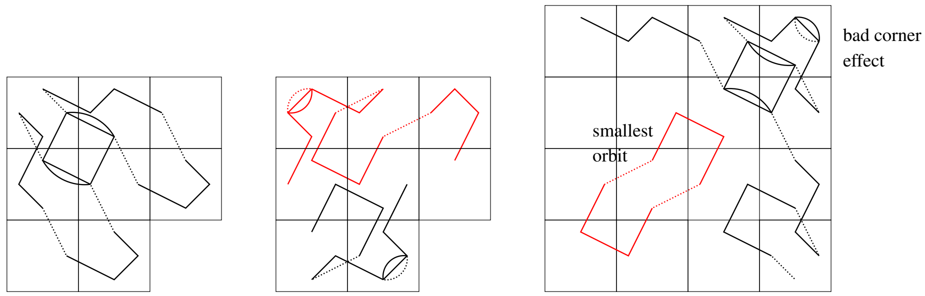

Figure 5: *Sample orbits for the variation on "Can of Worms" that repeatedly applies the sequence HSR, HSR, HSR, HSR^3. Solid lines indicate the HSR progression and dotted lines the HSR^3 progression.*

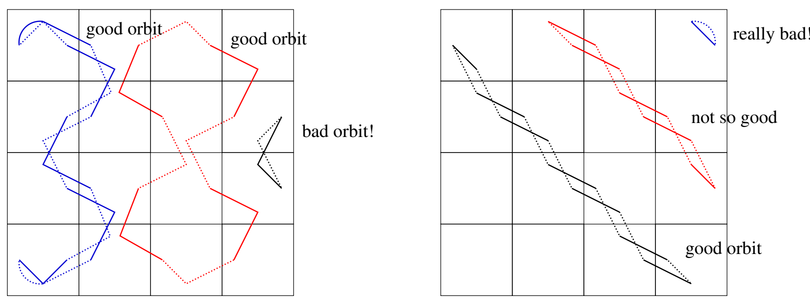

Figure 6: *Sample orbits for "Can of Worms," repeatedly applying the sequences HSR, HSR, HSR^3, HSR^3 (left) or HSR, HSR^3 (right).*

long-suffering couples are stuck in the corners of the grid for the entire dance—clearly, not a recipe for a successful evening!

5. Questions

This investigation has raised more questions than it answered. In a dance such as "Can of Worms," the caller is required to issue different instructions occasionally to avoid short orbits. One variation every four times through the dance is preferable to the other options we considered. Are there other good patterns? Does the size and shape of the grid matter much? We have noted that the amount of movement in "Warp and Weft" depends on the shape of the grid: a rectangular layout with dimensions that are relatively prime is ideal. Are there non-rectangular grids that have similar advantages? Are there certain grids that should be avoided? Do some layouts produce more mixing than others? Since each grid is different, the computer comes in handy. We have written MATLAB algorithms that allow us to visualize grid square progressions and generate statistics about mixing and movement. In addition to analyzing existing dances, there is potential that mathematics could assist callers in discovering new grid square progressions.

References

[1] Michael Buchler and Nancy Rogers. Square dance moves and twelve-tone operators: Isomorphisms and new transformational models. *Music Theory Online*, 9(4), 2003. http://www.societymusictheory.org/mto/issues/-mto.03.9.4/toc.9.4.html.

[2] Leon Harkleroad. *The Math behind the Music*. Outlooks. Cambridge University Press, Cambridge, 2006. With 1 CD-ROM (Windows, Macintosh and UNIX).

[3] Rick Mohr. Grid squares, 2011. http://rickmohr.net/Contra/Grids.asp. Retrieved Feb 1, 2011.

[4] Wing Mui. Connections between contra dancing and mathematics. *Journal of Mathematics and the Arts*, 4(1):13–20, 2010.

[5] Christine von Renesse and Volker Ecke. Mathematics and salsa dancing. *Journal of Mathematics and the Arts*, 5(1), 2011.