Symmetry Orbits: When Artists and Mathematicians Disagree

Darrah Chavey
Department of Mathematics & Computer Science
Beloit College
700 College St.
Beloit, WI, 53511, USA
E-mail: chavey@beloit.edu

Abstract

In many examples of repeating patterns in the art of various cultures, the use of symmetries to analyze those patterns does a good job of capturing the repetition intended by the artist. In other cases, however, the artist uses precise forms of repetition that are not well modeled by mathematical symmetries. The analysis of the orbits of motifs under the action of the symmetry group both reveals situations where this happens, and gives us direction as to what else is needed to model the artists' apparent intent. The ways in which the art of different cultures need different types of extensions of symmetry ideas reveals structural differences in the design art of those cultures, and hence in their ethnomathematics.

1. Introduction & Terminology

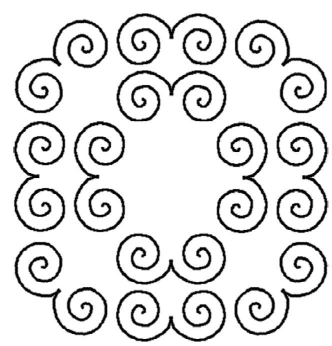


Figure 1: *A (simplified version of a) design of the White Hmong, with 3 orbits of the snail motif.*

In common usage, we can think of the "orbit" of a moon, or astronomical body, as all of the places to which that object moves under the forces of gravity. In adapting this terminology to symmetry groups, we say that an "orbit" of a motif is all of the copies of that motif to which it moves under the action (or "force") of the symmetry group. As an example, consider figure 1, a design of the White Hmong sub-group of the Hmong people of Vietnam, Laos, and China. Here the symmetry group of the design is the dihedral group D_4, i.e. rotations of ¼-th, along with reflections across four different lines (horizontal, vertical, and diagonal). Thus the 4 motifs on the inside of the design all lie in one orbit, while the outer ring of 8 motifs fall into two orbits: one on the main compass points, and the other on the diagonal points. This is also an example of why seeing such orbits has cultural, i.e. ethnomathematical, connections: The double spiral motif, referred to as the "snail" by Hmong artists, is used by many Hmong ethnic sub-groups. However, as best as we can tell, this use of central symmetry, with multiple orbits of the motif, occurs only in the art of the White Hmong sub-group. The Green Hmong also use this motif, but invariably use it in strip patterns and planar (wallpaper) patterns. Figure 3 (without the extra dashed lines) shows the snail motif as used on a dress created by a Green Hmong artist. This latter design has 1 orbit of the snail motif, while the former has 3 orbits. By traditional symmetry analysis, we might conclude that the Green Hmong use "more sophisticated" symmetry groups, but alternatively we could say that the White Hmong use "more sophisticated" multiple interlocking orbits.

When a design has only one orbit of an artistic motif, we suggest that the mathematical symmetry group of that design is generally a good model of the pattern repetition built into the design by the artist: Any repetition of the motif from the artist's viewpoint is modeled by a symmetry from the mathematician's viewpoint. Conversely, if a design has a large number of orbits of a motif, then it is likely that the symmetry-group analysis of the pattern gives an inappropriate description of the underlying artistic repetition. This happens, for example, with randomly placed motifs and with aperiodic tilings, e.g. [4, chapter 10]. Our focus in this paper is on intermediate situations: When a design has a modest number of motif orbits, more than one, and there is an artistic impression of those motifs being "equivalent." In this case, the symmetry-group approach appears to be capturing *some* of the design structure intended by the artist, but very likely has not captured *all* of that design structure. Here we want to identify where the symmetry-group approach has failed to capture the apparent artistic intent, and use variations of this approach to capture some of the apparent missing design principles of the artist or craftsman.

Although we do not discuss the topic in this paper, we should note that our previous statement, that a single orbit of a motif usually implies a successful mathematical model of the design, can fail when the motif itself has internal symmetries which do not extend to the pattern as a whole. Such designs could be analyzed using the techniques of this paper by artificially breaking those symmetric motifs into multiple pieces, i.e. by decomposing the motifs into their "fundamental regions."

Much of this paper consists of examples of specific types of cultural art where the symmetry-group approach to design analysis appears to need extensions of one form or another to fully capture the underlying design. As with the White Hmong/Green Hmong example, the ways in which the symmetry analysis needs to be extended often seems to be culturally specific, suggesting that these reflect culturally important artistic values and design principles of those cultures.

2. Double Bands vs. Wallpaper Patterns

A common design tactic that gives rise to multiple motif orbits occurs if we construct a single strip pattern by combining two copies of a smaller strip pattern. We call these "double-banded" patterns. The pattern of figure 2, from the Nazca culture in Peru (c. 500 C.E.), shows an example of such a design. Each of the strips alone has a p111 symmetry pattern, and they define a "perfectly colored" symmetry pattern on this design. However, the pattern as a whole has two orbits of the basic 3-step staircase motif, because there is no symmetry that moves the lower strip to the upper strip. The pattern has "apparent" translations along the indicated arrows, but applying those translations to the double-banded pattern would generate a 2-dimensional wallpaper pattern, which is *not* implied by the pattern itself.

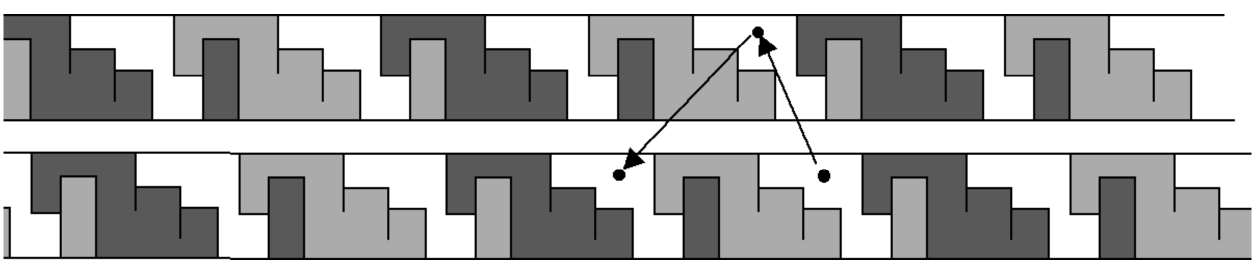


Figure 2: *A Nazca (pre-Incan) Peruvian design, re-drawn from* [6]*, p. 183, figure 274.*

Washburn and Crowe [8, p. 53] suggest that when a pattern has two adjacent strips, we should evaluate it as a 2-dimensional wallpaper pattern. However, their discussion of this point accompanies the analysis of pottery fragments, where it may be reasonable to infer the existence of a larger pattern from existing fragments. In designs such as figure 2, we know that this is the full extent of the pattern. If we choose to interpret this as a wallpaper pattern, at least three substantial problems arise:

1) Such an interpretation implies a design pattern with no evidence that this was the artist's intent;
2) How to extend a double-banded pattern to a larger wallpaper pattern may be ambiguous; and

3) We can extend two completely different double-banded patterns to get the same wallpaper pattern, implying that something important about the design has been lost in the process of this extension.

The Green Hmong dress pattern of figure 3 is an example of problem #3. Two different strip patterns are shown embedded within the design. If either of these strips were extended to a wallpaper pattern, we would, get the design shown here. And yet those two strip patterns are dramatically different. The horizontal pattern has a strip symmetry group of *pma2*, has one orbit under those symmetries, and the motif repetition is fully described by those symmetries. Conversely, the vertical strip has symmetry group *p1m1*, has two orbits of a "half-snail" motif, and the design repetition is *not* captured by the strip symmetry group without the use of additional descriptive tools.

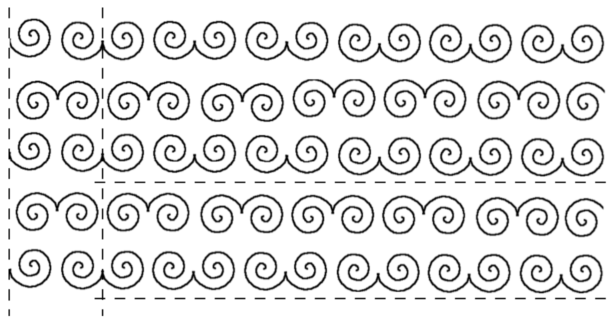


Figure 3: *A wallpaper planar design of the Green Hmong, demonstrating two different strip patterns within the design that can be viewed as generating the entire design.*

As an example of the second problem, the author has, on multiple occasions, used a worksheet in an Ethnomathematics course asking students to attempt to extend the pattern of figure 4a to a 3rd and 4th row, while maintaining the design of the first two rows. This design is re-drawn from a carving on the front of a Fijian canoe, and is another example of a design where we know that the artist carved only two rows. Consistently, about ¾-ths of the students extend the pattern as shown on the left in figure 4b, while the other ¼-th extends it as shown on the right. These two wallpaper patterns have different symmetry groups. Allowing color-reversing symmetries, the extension on the left has one orbit of triangles while the one on the right has two orbits. Since many people extend this pattern in each of these two "natural" ways, we conclude that presuming what the artist intended, even if *they* thought of it as part of a larger wallpaper pattern, is unwarranted.

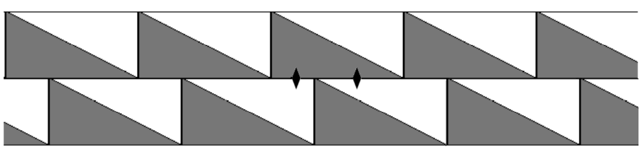


Figure 4a: *A carving from the front of a Fijian canoe. The design consists only of these two rows of triangles, and consists of 2 orbits of triangles. Black diamonds show rotation points.*

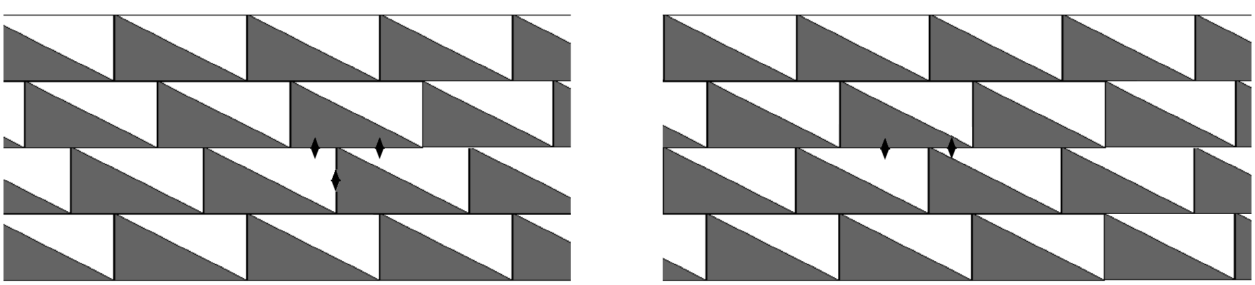


Figure 4b: *Two ways that students extend this design when asked to find a "natural" continuation.*

Instead of artificially extending a pattern such as those of figures 2 and 4a, we suggest that it is more natural to leave the pattern as it is, notice the symmetries which do exist, but also describe what additional actions would result in recognizing the equivalence of the different orbits in the two strips. This analysis is quite different in these two examples. The designs of figures 2 and 4a each have two orbits of motifs, but distributed quite differently. The Nazca design of figure 2 has all the motifs of the bottom strip in one orbit, and all those of the top strip in the second orbit. There is no symmetry that interchanges the two strips. But the Fijian canoe design has a color-reversing rotation that interchanges the two strips. Here one motif orbit contains the gray triangles of the bottom strip and the white triangles of the top strip, while the other orbit contains the remaining triangles. While we can describe the double-banded design of figure 4a using the "translate one strip" approach, a better approach is available, one that reveals "hidden symmetries" within the design. If we separate one of the two strips, as shown in figure 5, and look at the symmetries of that strip alone, the traditional symmetry analysis shows the equivalence of those two orbits of triangles. But the "translate one strip" approach by itself does *not* show the equivalence of these two orbits. To mathematically notice this design repetition within the pattern, we must use the "divide & conquer" approach of analyzing one strip in isolation.

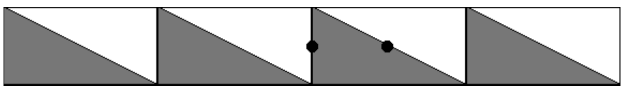


Figure 5: *A single strip of the Fijian canoe pattern has rotation points, at the black dots, which are not symmetries of the double-banded design.*

The examples of figures 3 and 4 indicate, as the first "problem" in our list suggested, that extending a double-banded design to a wallpaper design may be unjustified. In at least some cases, such an extension seems to be a clear violation of the artistic intent. Consider, for example, the American Indian basket of figure 6. In this double-banded pattern the two bands overlap each other. As with the Incan designs, we can view one band as a translate of the other. But if this *was* meant to be a portion of a wallpaper pattern, then portions of the band above the top one would have narrow points extending down into the visible design. The artist appears to be deliberate in *not* viewing this as part of a larger design.

Figure 6: *An American Indian basket from northern California, possibly Hupa origin, early 20th century. Photo used courtesy of the Logan Museum of Anthropology, Beloit College.*

The use of 2-orbit, double-banded patterns are common, but not exceedingly so. So it is an impressive cultural phenomena to see the persistence of these designs across centuries of Peruvian culture. Our references show many such examples in Peru from the 15th century [1, pp. 176-179], 14 examples from the 6th–8th century [3], 16 examples from the 4th–6th century [6], and 4 examples dating as far back as the 5th century B.C.E. [7]. These examples also show a persistence in how their two strips are related. In such patterns, one band can be a copy of the other via any of the following general actions:

1) A translation of one strip to generate the second (see figures 2 and 6);
2) A translation followed by a vertical reflection, i.e. a glide reflection (not shown here);
3) A mirror image, i.e. a horizontal reflection (see figure 7);
4) A rotation of the bottom strip to the top (see figure 4a and 8); or
5) Some combination of the above.

So it is impressive that of the 34 early examples from [7], [6], and [3], 23 are of type (1), 10 are of type (2), and only 1 is of any other type. This contrasts dramatically with the double-banded patterns of Papua New Guinea, in the following section. The examples we show there all fit into categories 2–5, and of all of the examples we have seen from this region, we are unfamiliar with *any* examples of double-banded strips of category 1. While we do not discuss possible reasons for these differences here, we believe that this demonstrates a clear cultural difference in these two cultures' use of symmetry in artistic design work.

3. Art of Papua New Guinea

Double-banded designs are also common in the art of Papua New Guinea. Figure 7 shows several examples of such designs. The objects here all have mirror reflections between the two halves of the design. In each case, copies of the main motif fall into two orbits. The equivalence of the two motif orbits can be modeled by slicing the pattern in half, and using the symmetries of a single strip to connect those orbits. As with the Incan double-banded patterns, extra strip symmetries do *not* extend to the entire design. With the paint board (7c) and the gope board (7d) these "hidden" symmetries are color-reversing. With the gope board, that hidden symmetry is a rotation that interchanges the red wave motifs with light brown wave motifs. In all other cases, there is a glide reflection that interchanges the motifs on the "inside" of the pattern with the motifs on the "outside" of that pattern.

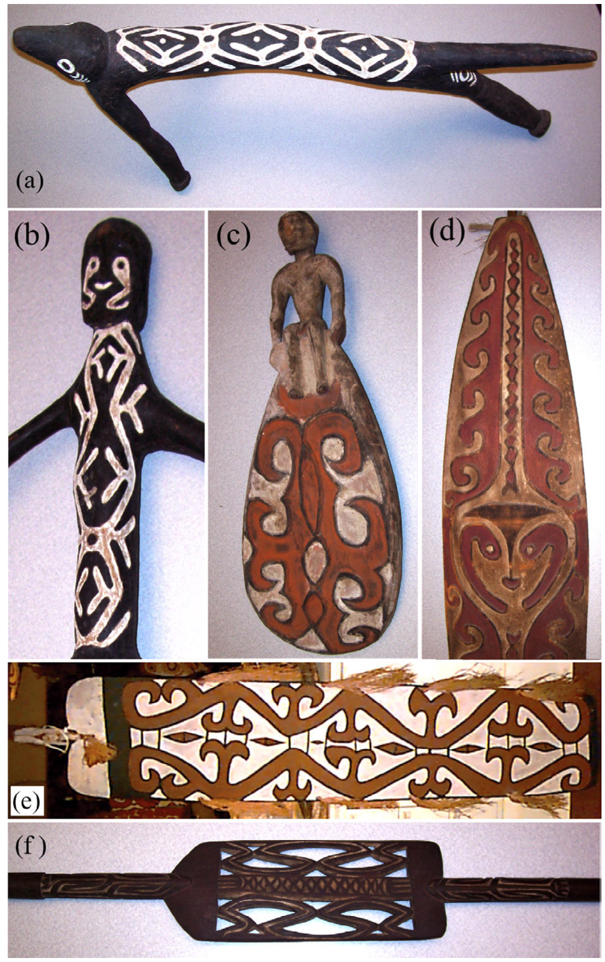


Figure 7: *Papua New Guinea art: (a) 3-legged dog; (b) Closeup of a man; (c) Paint board; (d) Gope board; (e) War shield; (f) Wings on a spear. The war shield is used courtesy of the Logan Museum of Anthropology, Beloit College. All other objects are in the Ethnomathematics collection of the Mathematics Department, Beloit College. (Objects from Papua Gulf and the Asmat region.)*

There is a significant similarity in the motifs of these designs. Both the three-legged dog (7a) and the man (7b) use the same "Y" shaped motif, which alternates between pointing in and pointing out. The war shield (7e) and the paint board (7c) use the same double-leaf motif, which is closely related to the "Y" motif. Nevertheless, the similarity across the mathematical structures of these designs is even stronger than the similarity in motifs. Instead of one strip being a translated version of the other, as in our previous examples, here one strip is a mirror image of the other. Each individual strip contains two orbits of the motif (with respect to the symmetries of the full design), and there is a glide reflection within an individual strip that "reveals" the equivalence of those two orbits. The fact that this particular mathematical structure arises so frequently in Papua New Guinea, and infrequently in other cultures with which we are familiar, indicates that this nested "pattern inside a pattern" structure reflects an artistic tradition of Papua New Guinea. The three-legged dog (7a) gives interesting evidence that our approach of separating a single band from the double-banded pattern for analysis may correctly reflect the artistic intent. The photo here is taken looking from above, so that most of the pattern is visible. But when viewed straight on, with the dog's back at eye level, what is visible is exactly one of the strips!

Further evidence that this "pattern within a pattern" design structure is deliberate comes from other Papua New Guinea artifacts with additional levels of patterning, such as the horn on the left in figure 8. If we were to unroll this design from the cylindrical horn, we would get the design shown on its right. Here the "tree branches" have a mathematical structure much like the artifacts of figure 7: a translation and rotation in the full pattern, with two orbits of tree branches whose mathematical equivalence is seen only by separating one strip from the other. That analysis would then recognize the design equivalence of all the branches, but not of the leaves along the branches. Those leaves are connected by a glide-reflection acting on a single branch. Thus if we use the "traditional" symmetry-group analysis of this design, we see 6 orbits of the leaf motif. If we allow the additional rotation symmetries within a single strip, we "see" that there are only three orbits of leafs (A+a, B+b, and C+c). But, only if we allow ourselves to apply symmetry groups to the pattern of the leaves along an individual branch, does the mathematics "understand" the equivalence of the leaves, which appears to be the artistic intent.

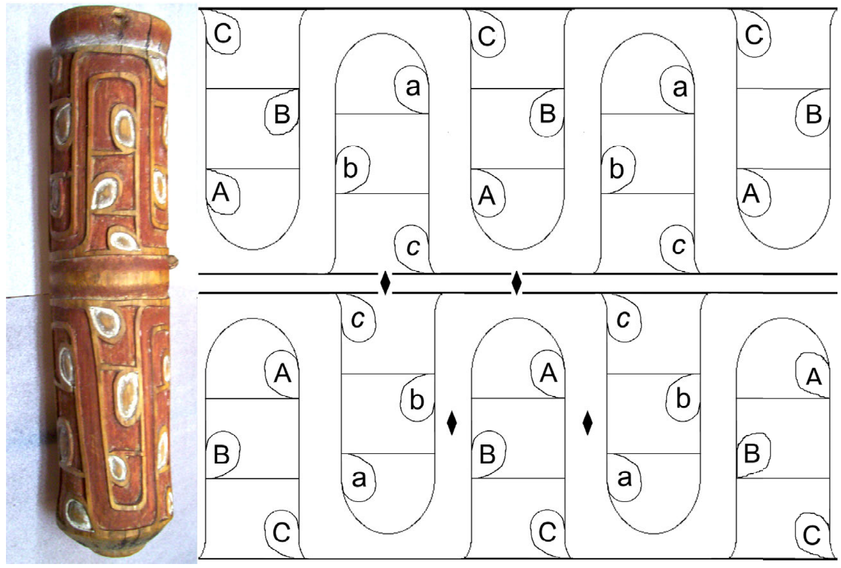


Figure 8: *A horn from Papua New Guinea, along with a schematic showing the "unrolled pattern". Rotations of the full double-banded pattern and of the lower strip in isolation are indicated by diamonds. From the Ethnomathematics collection of the Mathematics Department, Beloit College.*

4. Maori Rafter Patterns

Maori rafter patterns are a well-studied example of interesting and complicated strip patterns, e.g. [1, pp. 166-172]. The approach of this paper can be used to re-discover some of the design aspects of these patterns. Three examples of Maori rafter patterns are shown in figure 9 (these are each sections of longer strip patterns). With the top two patterns, the combination of color-preserving and color-reversing symmetries "sees" a design with two orbits of a broadleaf motif shape. If not for the circled symmetry-disrupting features, a "baby fern" (or "scroll") motif, these broadleafs *would* fall into a single orbit. This is a very common feature in Maori art: the baby fern is often used to disrupt symmetries of a larger pattern. As others have noted, a full description of the symmetry thus requires a description of the pattern both with those ferns in place and with them removed. The orbit analysis of this paper leads to the same conclusions: There are two orbits of the leaf motif using standard symmetry analysis; and some additional descriptive tool is necessary to mathematically "notice" the equivalence of those orbits. The bottom left figure shows another Maori rafter pattern with several of these baby ferns. If we remove those ferns, we discover a set of "wave" motifs shown on the right, in two orbits, whose artistic and mathematical structure is nearly identical to the Papua New Guinea gope board pattern of figure 7(d)!

Figure 9: *Three Maori rafter patterns (from Hamilton, via WikiMedia Commons).*

5. Afghani Dress Flowers

There are two common types of symmetric 3-colored patterns: Those with one third of the pattern in each color, and those with the colors distributed ½, ¼, and ¼, i.e. in ratios 2 : 1 : 1. In this later case, we expect there to be symmetries which fix the dominant color while interchanging the two minor colors, and symmetries which interchange the major color with the union of the two minor colors. An example of such a color structure is shown in the beaded "dress flower" of figure 10, from the Pashtun region of Afghanistan. To its right, we've indicated the original colors of the 8 motifs in this design.

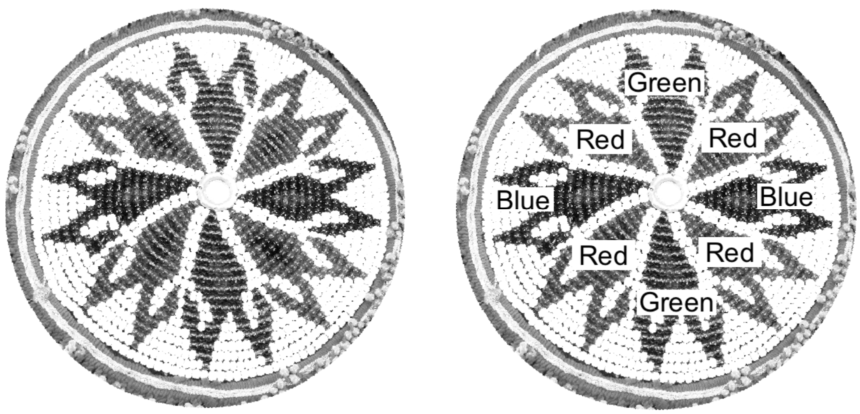
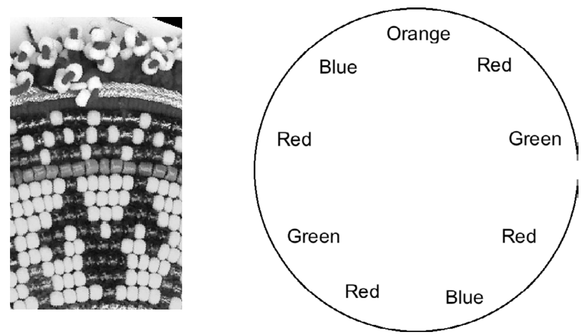


Figure 10: *An Afghani dress flower of the Pashtun region with motif colors in the ratio 2:1:1 (Red : Green : Blue). From the author's personal collection.*

The standard approach to analyzing color symmetries (e.g. [4, chapter 8]) is to require each symmetry to permute the colors. With this constraint, the dress flower here would have two orbits of motifs: The red in one orbit, and the green and the blue in the other orbit. Thus to capture the apparent artistic intent of the equivalence of the sets {Red} and {Green, Blue} we must broaden our concept of colored symmetries to allow for symmetries that interchange one *set* of colors with another *set*. The cultural examples with which we are familiar involve only this one extension: a 3-color pattern where we allow a "symmetry" that interchanges the dominant color with a union of the other colors.

However, several of these Pashtun (Afghani) dress flowers contain an additional symmetry-disrupting element. We have, or have seen, several examples of such dress flowers where *most* of the motifs would have a color symmetry group like that of figure 10, but a single additional motif disrupts this symmetry. For example, the Afghan dress flower shown in figure 11 has a single orange "M" motif that dramatically disrupts the overall design repetition. If the orange motif was removed, and the others re-aligned, we would have a standard, 3-color, 2:1:1 design. But with the orange "M", the only symmetry is the right-to-left reflection, leaving us with 5 orbits of the motif. Again, this is a signal that the symmetry analysis has *not* captured the design intent of the artist. That this is artistic intent, and not accidental, seems to be implied by the dozen or more examples we have of such dress flowers with this very specific symmetry disruption, and the use of a special, otherwise unused, color for the one disrupting motif.

Figure 11: *A beaded dress flower from the Pashtun region of Afghanistan. The "*M*" motif shown on the right is repeated multiple times around the circle (7, 9, 11, and 13 times in various examples we have), beaded in different colors. Each of these examples has an overall color-regular pattern disrupted by a single "*M*" of a contrasting color.*



6. Some Related Work

Many authors have investigated aspects of designs similar to those described here. We mention only a few particularly relevant sources. Washburn & Crowe [8, chapter 7] discuss in detail "Problems in Classification." This includes discussion of examples, similar to the Maori art and Afghani dress flowers, where it may be appropriate to ignore some symmetry-disrupting elements. They also discuss "compound patterns," where the appropriate analysis may be to separate a design into two (or more) component pieces, each analyzed separately. These compound patterns may be "layered," such as with a background pattern of one symmetry type, and a foreground pattern of another. Bérczi [2] is an analysis of a large corpus of such compound designs from Eurasia. These patterns are another category of designs where our "orbit analysis" would reveal the necessity of symmetry analysis beyond the traditional symmetry groups. Of course this is an example where such deeper analysis was already recognized.

7. Conclusions

In analyzing the pattern designs of cultures, the identification of the symmetry groups often does a thorough job of capturing the design repetition that the artist has embedded in their art. In some cases, however, the mathematical symmetry group does not capture the full design repetition, and hence may fail to capture the full design intent of the artist. Many cultures find the tension between regularity and its disruption particularly appealing, and express that tension in various ways.

To find the symmetries within a design, we look for symmetries that move motifs to each other, and look at symmetries of a motif that extend to symmetries of the entire design. In doing so, we can determine if the formal symmetries of a design result in all motifs lying in the same orbit. When this does *not* happen is, we claim, when the mathematical-artistic analysis gets interesting. While this may mean that symmetry groups are an inappropriate tool for analyzing that design, we argue that in many cases this means, instead, that we need to be more creative in our use of symmetry analysis. In these cases, modification of the mathematical approach can complete the modeling of the design repetition. The particular type of modification(s) that are required are often specific to a particular culture, and the culture-specific aspect of their designs is an insight into the ethnomathematics of that culture. In some cases discussed here, we believe we may have insight as to the reasons for the use of specific symmetry variations; in other cases the art requires more study to find such reasons. But it is the symmetry analysis that raises the questions.

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