Three Approaches to Regular Linked Structures

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Abstract

A dedicated geometer, a magic woodcarver and an origami master go treasure hunting on the same mountain. Many of the treasures they dig out are similar, but others bear witness to the different interests that motivated their search. Being, perhaps, the least picky of the three, I find I can use what one of the others throws out.

Cast

After working in solitude for many years, I have begun to feel more and more like a link in a chain. In this paper I want to explore how my work relates to that of the former Bell Labs scientist Alan Holden and the professional origami master Dr. Robert J. Lang. I consider them closest links to me in this chain. Our work contains many similarities, but also differences that reflect the different interests that brought us to work on the subject. The chain metaphor has more to do with the subject matter than with the relationships between us. We have all worked independently: I did not hear about Holden's work until I had done most of the work that resembles his; Lang did not hear about Holden until he presented his own work at the Gathering4Gardner in 1999; I did not hear about Lang until I presented my own work at Bridges 2010; and I don't think that either of the two have heard about me, or Holden about Lang.

At the Bridges conference 2010 I presented myself as a magic woodcarver and explained a few of my best geometric techniques for developing new models for carving. Starting in 1965, my first attempts in the art of magic woodcarving produced traditional pieces like chains and balls in cages, but I soon became interested in developing more sophisticated models. Among my first successes were some constellations of regular polygons woven together in symmetric patterns. Next I looked for non-flat structures, such as polyhedral frames and hosohedra, that could likewise be woven into symmetric compounds. Later I developed curled and even knotted structures that fit together symmetrically.

When Alan Holden's book *Orderly Tangles* [2] came out in 1982 I was amazed to find all of my polygonal models plus many more of the same sort. Holden had done a systematic search for such structures using

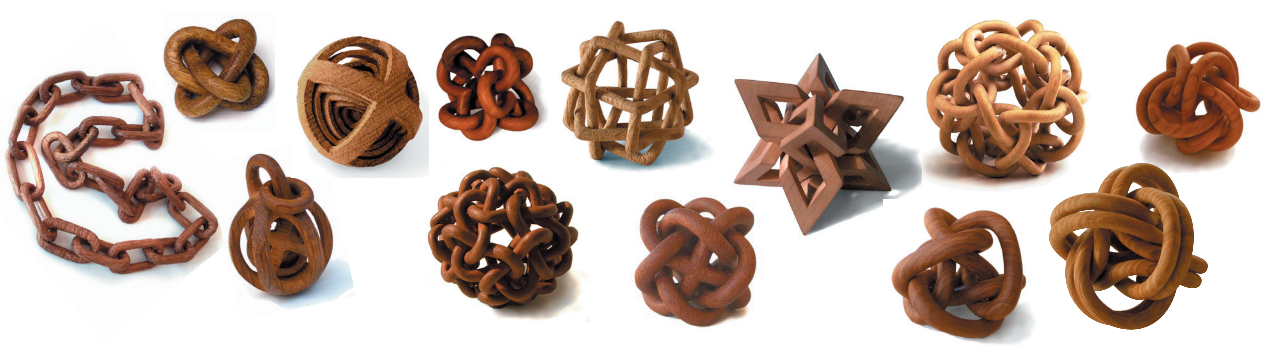

Figure 1: *This retrospective selection show a gradual increase in the complexity of my carvings.*

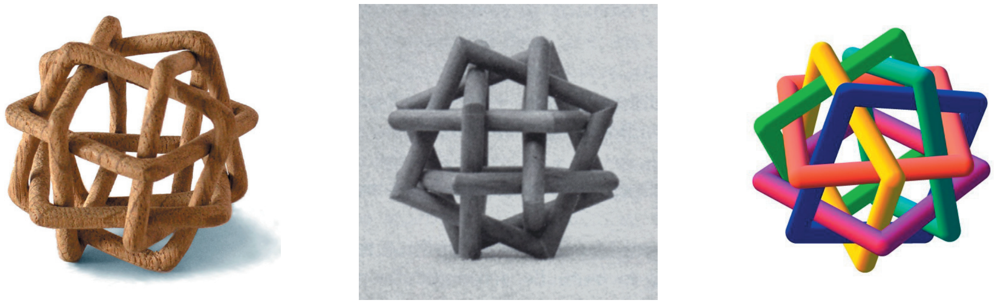

Figure 2: *The same structure realized as a magic woodcarving, a regular polylink in dowel, and a virtual model of a polypolyhedron (by Jespersen, Holden, and Lang).*

practical methods such as plaiting cardboard strips resting on polyhedral scaffolds. He displayed his findings as precisely crafted dowel models. Holden's motivation seems to have been a deep fascination with the beauty of regular solids and their many interesting properties, and a desire to display this beauty in skillfully crafted models. This fascination is indeed vividly and captivatingly exposed in his earlier book *Shapes, Space, and Symmetry* [3], which had been one of my treasured sources of inspiration since it came out in 1971.

Robert Lang's work was unknown to me until I submitted last year's Bridges paper for review, and the reviewer advised me to look at it. I did [7] and was impressed! Lang's project is very bold: he defines a class of structures which he calls polypolyhedra and claims to have found all of them – a total of 54. The class he defines is motivated by his desire to find candidates for a special type of origami models. Lang was impressed by another origami master, Thomas Hull, who had turned the familiar five intersecting tetrahedra into a perfect origami model (Figure 10). Lang wondered what other linked structures could be found that could be turned into similar origami models. What followed was a genuine mathematical tour de force.

Definitions

The classes of structures studied by the three of us are overlapping but differ according to our underlying interests. Alan Holden was interested in and built models of various types of geometric objects – knots, weaving patterns, cat's cradles, etc. There is no doubt that he had a special interest in the class of structures I shall deal with here, which he called *regular polylinks*. His precise definition was ... *[a structure] made of rings shaped as regular polygons whose corners are indistinguishable in the structure.* Holden, in other words, required vertex-regularity and limited his search to flat rings. He worked systematically to find all such structures, but, as he was well aware, his method offered no guarantee of completeness, and he suspected – rightly, as we shall see – that he had missed some.

Robert Lang's purpose was to search for structures that could be turned into modular origami models. Each edge of the structure would correspond to a modular unit, and there should be just one type of module; in other words, all edges should be alike. So Lang requires edge-regularity, which is weaker than vertex-regularity since it allows for two types of vertices, each edge having one end of either type. His precise definition reads like this … *a compound of multiple linked polyhedral skeletons with uniform nonintersecting edges.* In Lang's terminology a polygon is considered to be a polyhedron with just one face. Since vertex-regularity implies edge-regularity, this means that all of Holden's regular polylinks should be included (as well as the ones he did not find).

As a woodcarver, I have no special preference for straight lines. Curved lines are in fact easier to work with, as no one will notice a slight difference in the curvature of two curved lines, whereas anyone will notice the slightest curvature in a line that is supposed to be straight. Curved lines also have aesthetic qualities that I often prefer. I have never attempted to set up a formal definition for the class of structures I am looking for – I guess my only criterion is that *the parts should be identical and symmetrically linked together*. We might refer to this as component-regularity. It is a much weaker criterion than either Holden's or Lang's. Consequently the class of possible structures is much larger and the question of enumeration does not arise. After all, when you allow the parts to curl around each other and even tie knots on themselves, the possibilities are unlimited. You could try to look for all possible structures with a limited number of crossing points, which is the way knots are normally tabulated. That's not what I have been doing, however. I am merely looking for beautiful models to carve.

Methods of construction

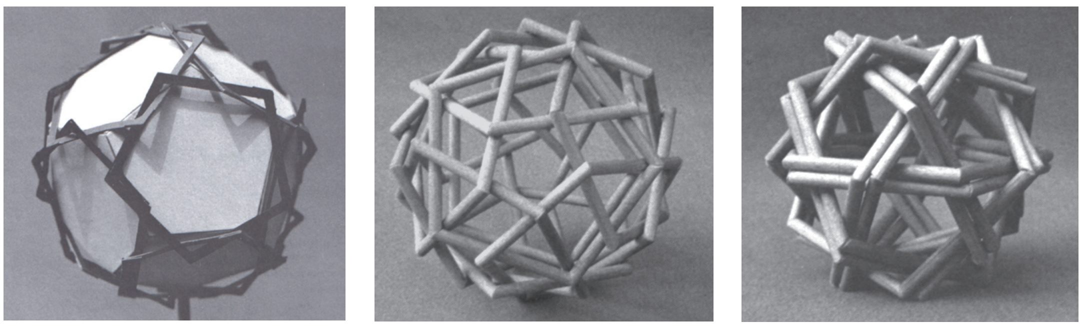

Figure 3: *Characteristic steps in Holden's method of construction (copied from his book).*

Alan Holden's approach was very practical. 3D computer software was not really an option in 1980. He argued on the basis of regularity that each polylink could be derived from one of the Platonic solids. Imagine one of the five, e.g. the cube, and think of the perimeters of its faces as the rings of a polylink. They are not yet linked; rather, their edges coincide with those of their neighbors. Now rotate each ring slightly around its center and it will immediately link up with each of its neighbors. If you shrink the imaginary cube, thus moving the rings closer to the center, and at the same time rotate them a little more, the rings will engage each other in a new way (Figure 2). Note that this imaginary process involves letting the rings pass through each other. In the case of the cube, you will find only these two configurations. But Holden, of course, performed similar operations on all five Platonic solids.

Moving on to the octahedron, dodecahedron, and icosahedron, the situation becomes more interesting. The number of possible configurations grows larger. The number of rings linked to each individual ring grows larger, as the rings move closer to the centre. In the end, you reach a configuration in which opposite rings come together and touch one another (Figure 3). Fusing each pair of rings in this configuration into a single ring gives you yet another regular polylink with half as many rings.

Robert Lang's method is more sophisticated. He found the mathematical tools needed to ensure that he would find all structures satisfying his definition. Just like Holden, he focuses on symmetry. Edge-regularity implies that all edges can be derived from a single reference edge by applying the rotational transformations of some symmetry group. Although there are five Platonic solids, they only represent three symmetry groups. They are named after their triangular representatives: the tetrahedral, the octahedral, and the icosahedral symmetry groups (T, O, and I). Lang also considers cyclic and dihedral symmetry groups, but only to demonstrate they contribute nothing. Consider the reference edge again as it is transformed by every rotation in the relevant symmetry group. The images of these transformations will make up the entire polypolyhedron. The endpoints of the reference edge will of course be transformed into the vertices of the polypolyhedron. These may be all of one type (if the polypolyhedron is vertex-regular) or of two types if the reference edge (and every other edge) connects two different types of vertices.

The set of all images of an object created by the rotations in a symmetry group is called an orbit. For an arbitrary point, the number of points in its orbit is the order of the rotation group (i.e. the number of different rotations). Such an orbit is called complete, and labeled C. Points lying on one of the rotation axes have smaller

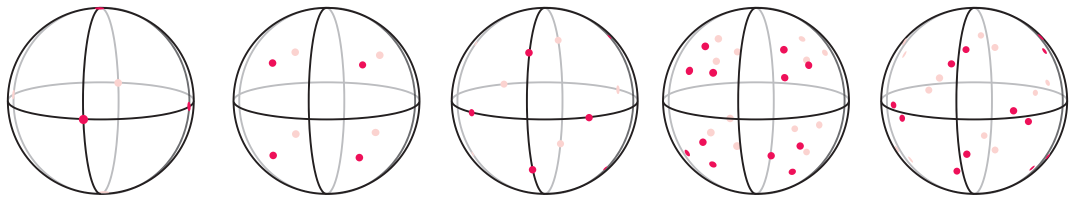

Figure 4: *Orbits play a key role in Lang's construction. These are the three lower order orbits and two examples of complete orbits in the octahedral rotation group. The order of each is shown in parenthesis.*

orbits (because they are not affected by rotations around that axis). Since each symmetry group has three types of rotation axes – through vertices, midpoints of faces, and midpoints of edges of the associated polyhedron – we have three more types of orbits for each of the three symmetry groups, which we will label V, F, and E (Figure 4). Now an edge of any edge-regular polypolyhedron will either have both end points in the same orbit or it will have an end point in each of two orbits in the same symmetry group. In this way Lang transforms the task of finding all edge-regular polypolyhedra into the task of testing all such edges. This is not a small task, but it turns out to be manageable. The technique is guaranteed to produce all edge-regular polypolyhedra, but it will also produce many other structures that will not qualify: some will have edges with free ends; some will have edges that intersect; and some will be a single connected structure rather than linked substructures.

Lang classifies edges according to the types of orbit to which their endpoints belong. He presents some clever arguments to rule out many combinations, and ends up with just ten types of edges - four of them homo-orbital, i.e. with both ends in the same orbit: VV, FF, EE, CC, and six hetero-orbital i.e. with their two ends in different orbits: VF, VE, FE, Vv, Ff, Ee (note that it may be two orbits of the same type but with different radii). He then sets up representative orbits for each of the three symmetry groups (T, O, and I) and searches for edges of the ten relevant types. He determines there are 188 distinctive edges to consider. Every structure derived by subjecting one of these edges to every rotation in the relevant symmetry group is guaranteed to be edge-regular, but not to meet the other criteria he had set up for interesting polypolyhedra. Actually only 41 of them qualify. (We shall soon see that structures falling short here may still be very interesting!)

The 41 structures, however, are not the end products, since they are based on representative orbits. There is a certain degree of randomness in the choice of seed point for each orbit. For the lower order orbits (those lying at the rotation axes) the only choice is the radial distance from the center. Complete orbits also require a two dimensional position away from the symmetry axes. To see what happens when these parameters vary continuously, Lang uses minimal distance functions that drop to zero whenever two edges intersect. He argues that the intervals (or areas) between such zero points (or curves) represent topologically different structures. This final analysis brings the total number of polypolyhedra up to 54. An interesting point can be made about the way Lang treats the CC-structures (those whose edges connect two points in the same complete orbit). He observes that they all consist of regular polygons and can be classified according to the type and number of polygons. Within each class, all possible constellations of the polygons are found by varying the two parameters defining the orbit. Lang argues that this variation is equivalent to a combined rotation and radial translation of the polygons – exactly the same transformation that Alan Holden used in his experimental search for these structures. By far the largest of these classes is the one featuring structures with twenty equilateral triangles. These are the structures of which Alan Holden wrote: *Until now twelve such polylinks have been discovered, and no doubt more will follow.* He was quite right: the correct total number is 23.

My own methods of construction are much more diverse, since I am not looking for a well defined type of objects. At last years Bridges conference I presented two of my most productive geometric techniques, one of them being the use of rhombic solids to record the weaving patterns for families of related structures. Now I will demonstrate a use of this technique that is relevant to our present context. To sum up the technique:I use three rhombic solids to represent the three spatial symmetry groups: a cube for the tetrahedral group (its faces can be considered right angled rhombi), a rhombic dodecahedron for the octahedral group, and a rhombic triacontahedron for the icosahedral group. A base module for a weaving pattern can be recorded from a known

Figure 5: *The results of applying a single rhombic diagram.*

structure or sketched from intuition. I mark it up on a rhombic unit cell, which I copy to all faces of the rhombic dodecahedron and the rhombic triacontahedron. Then I squeeze the pattern into a square that I copy to the faces of the cube. Finally I squeeze it again, exchanging the roles of long and short diagonal, and copy it to all faces of the rhombic dodecahedron and triacontahedron once more (Figure 5). Maybe you will better understand the logic behind this trick if you imagine the rhombic unit cell with a single diagonal (rather than a complicated weaving pattern). The result would simply be the edge frames of the five Platonic solids. The beauty of this technique is that any interesting idea is potentially multiplied by five. Although the basic weaving patterns are the same on all five solids, the change in symmetry may create significant differences in the overall patterns. The resulting structures may not always be suitable candidates for magic woodcarving. It may happen that one is a compound of interlinked substructures while the other four are single connected structures. More interesting phenomena also occur: e.g. a pattern forming simple linked loops on a large rhombic solid may form knotted loops on a smaller solid. This is the case with the example I am going to explain.

I was working with linked knots and had found three different tetraknots that all had their component knots offset from the center much like the faces of a tetrahedron (Figure 6). The tetrahedral faces, however, can be moved in towards the center until their individual centers coincide, forming a familiar link of four triangles (Figure 7). Hence I thought it was natural to look for a tetraknot with concentric components. A simple way to construct one is to transform each of the four linked triangles into a trefoil knot. This transformation can be performed at the level of the rhombic unit cell (Figure 8). Turning the triangles into knots, however, does not significantly change the simple way they are linked. I wanted the knots to be more open and mutually entangled. As you open up the knots, their strands will soon collide at the three- and fourfold symmetry axes. Imagine them being able to pass through each other (like ghosts through a wall) beyond the symmetry axes to open the knots even further. Again this transformation can be performed at the unit cell level (this takes some practice). The resulting structure is the Great Concentric Tetraknot seen in Figure 9.

This is all very useful, but the real surprise comes when you apply the same pattern to the rhombic triacontahedron (rather than the rhombic dodecahedron). This produces a structure of 15 rings linked together in an interesting way: The 15 rings can be grouped in five sets of three forming Borromean links (Figure 16).

Figure 6: *Three tetraknots with eccentric components.*

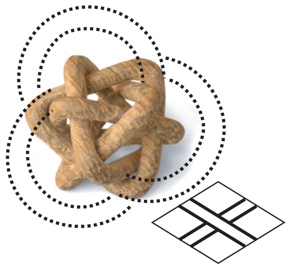

Figure 7: *A simple tetralink may be transfomed into a tetraknot. For this we will use the weaving pattern.*

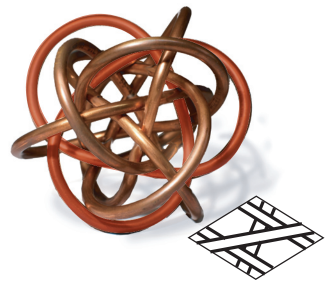

Figure 8: *To turn the triangles into knots, doubble the strands and make them cross at each of the original undercrossings.*

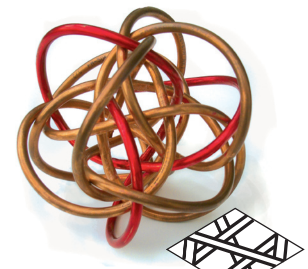

Figure 9: *Moving the strands of a weaving pattern past the symmetry axes takes some practice. Here is the result.*

Comparisons

One of my initial goals with this study was simply to verify that Lang's polypolyhedra really included all of Holden's regular polylinks, as they should if Lang's claims are valid. At my first review of Lang's paper I did not fully grasp everything, and I admit I was skeptical. After a while I reread it, this time with a real effort to understand. That removed my skepticism – this was rigorous mathematics. I had no more doubts that all of Holden's models would be there. Yet I decided to record the rhombic weaving patterns for all 26 relevant models in Holden's book and all 54 models featured on Lang's web site. This proved to be more difficult than I had expected. Some pictures seemed like a mess, and without the ability to turn the model around it seemed impossible to extract the information I needed. But I managed in the end, and could confirm that Holden, as he expected, had found all he was looking for, except in the last and largest of his categories derived from the icosahedron. A strong motivation to complete this tedious work was that these patterns could provide a rich source of material to work with. Who knows what wonders will turn up when I run them through my rhombic kaleidoscope?

Holden singled out three models of concentric, equatorial rings as *the crown of the regular polylinks*. They are the ones that he arrived at by melting together the pairs of parallel rings that are the end products of his search procedure when applied to the octahedron, the dodecahedron, and the icosahedron. Lang classifies these three together with the five intersecting tetrahedra (FIT) as non-CC homo-orbital. The reason Holden did not include the FIT is not that he was unaware of it, but simply that it does not consist of flat rings and therefore is not a regular polylink by his definition.

It is interesting that Holden's procedure, when applied to the cube, does not end in pairs of rings that can be melted together. This is where intuition might expect to find the Borromean rings, but alas, the square rings will not fit symmetrically like that. Nevertheless Holden at this point, without explanation, introduces two

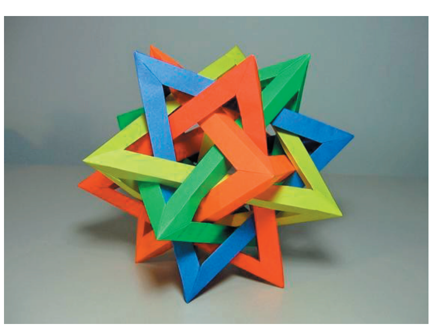

Figure 10 and 11: *Hull's FIT an Lang's FIX - the initial inspiration and the pride of Lang's remarkable effort.*

versions of the Borromean rings, one with regular hexagons and one with hexagonal stars. He simply remarks that they are not regular as they have two types of vertices. No one, by the way, seems to have considered the double Borromean rings. True – they would not be edge-regular but still, the two types of edges would be each other's mirror images, so an origami model could still be made of identical pieces, just folded oppositely.

Lang also has a favorite among his 54 models: a constellation of five interlocking hexahedra (Figure 11). He singles it out as one of two to have no 2-valent vertices. The other one is the one with five intersecting tetrahedra. This is known among origami enthusiasts as Hull's FIT; I suggest the former be known as Lang's FIX (X standing for hexahedra). Lang himself dubbed it Chomolungma, thinking it would be easy to remember - well, maybe yes, if you are a mountain climber! He has every reason to be proud of this one. I find it quite embarrassing not to have found it myself. I can see several ways by which I could have discovered it. But I didn't!

I also have my favorites, and it turns out that some of them are closely related to structures that Lang discards in the "elimination race" to his final 54. He found 188 different edges that would lead to different edge-regular structures. He ciscarded most of them, however, because they failed on one or more criteria he had established for a structure to qualify as a proper polypolyhedron. One of these criteria is that there must be no intersection of edges.

Let us look at some of this "scrap": Consider the two tetrahedral frames in Figure 12 (an O/FF structure). Lang discards these because their edges intersect. I also discard them, but for a very different reason. Since I do not require straight edges, I would first look for ways to bend the edges to avoid the intersections. In this case,

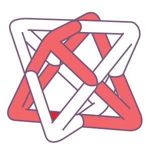

Figure 12: *Two tetrahedral frames with edge crossing that cannot be resolved.*

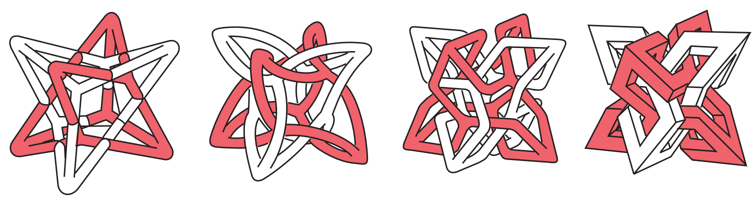

Figure 13: *Two hexahedral frames whose crossings resolve beautifully.*

however, it is not possible to do this in a regular fashion so that the two components are identical. That is why I also discard this structure. Now look at another example (an O/Ff): the two hexahedral frames in Figure 13. Lang discards them without a second thought, whereas I immediately recognize them as the basis of one of my best models: Bend the edges slightly to avoid the intersections. Then give each edge a kink in the middle so that every half-edge runs parallel to an edge of the tetrahedron defined by the outer vertices. Given the right cross section, these edges will always meet face to face, parallel to the tetrahedral faces. I wonder if someone could turn this into a modular origami model. I carved it from a block of pear wood (Figure 1).

Early in my career I devised two closely related hosohedral links that I call Flexus and Asalink. They have the same basic weaving pattern, but applied with different orientation to the rhombic dodecahedron. Lang's work duplicates the one but discards the other because it requires bent edges (Figure 14). A more complicated

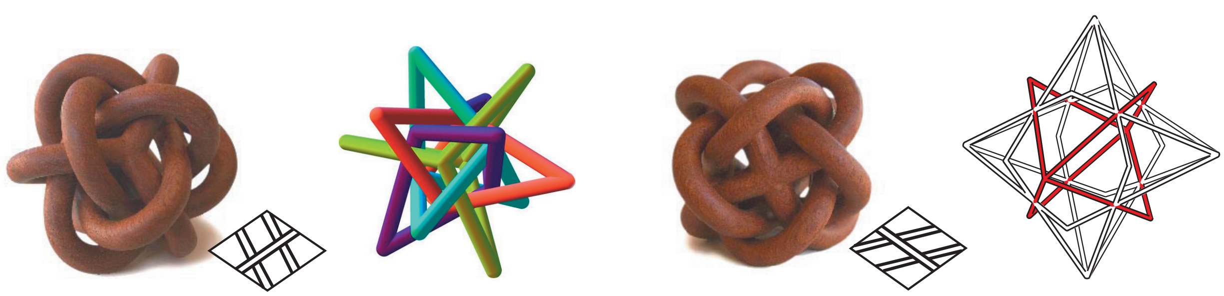

Figure 14: *My Flexus (left) is duplicated in Lang's work as no. 10. Its close relative Asalink, however, is not to be found, since the straight-edge version contains edge crossings.*

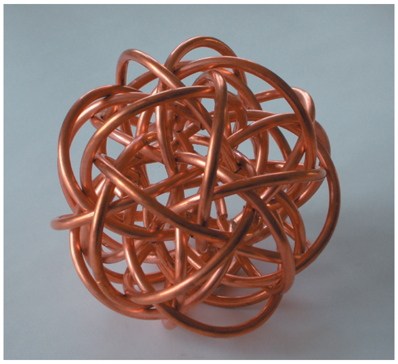

Figure 15: *Copper model of the Quintuble Borromeo the way I intend to carve it.*

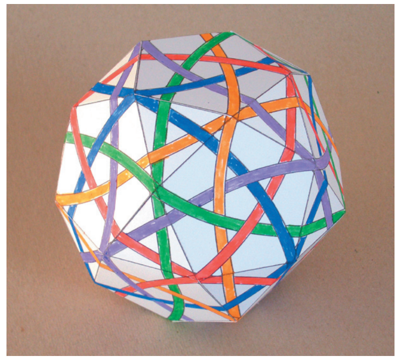

Figure 16: *The coloring shows that the fifteen rings form five sets of Borromean rings.*

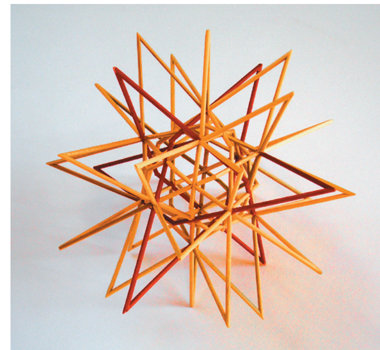

Figure 17: *The three colored rhombi form a Borromean link in this I/Ee structure.*

example is the one I used as example of my construction methods: the Quintuple Borromeo seen in Figure 15. This too has a counterpart swept out from Lang's workshop because the straight edges intersect at both three- and fivefold axes (Figure 17). Thus it seems to me that Lang is throwing out a lot of beautiful babies with the bathwater. Something he would not have done if he had been a woodcarver rather than an origami master. But then he would not have found his beautiful proof of the completeness of his list of the 54 polypolyhedra.

There is something odd about the way objects like these enter the stage. Think of Lang's FIX or my Hexacoil, for example (number 4 in Figure 5). Presumably, no one had ever seen or thought of them before we introduced them. Does this mean that we created or invented them? Yes and no. What we invented was the technique that led us to discover them. It feels like they were already there, just waiting to be found. This, of course, has no meaning, unless you are a hardcore Platonist.

My account of Holden's and Lang's work is no more than a brief summary. Readers who want more detail should consult the original sources listed below. My own work is as yet unpublished except [5]. However, a book is scheduled to be published in the spring of 2012 by Fox Chapel Publishing Company; working titles: *Magic Woodcarving or Woodcarving Wizardry.*

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