Good Stamps for Wallpaper Patterns

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Abstract

The problem of identifying how many (doubly periodic) patterns exist on the plane, which are different from the point of view of symmetry can be approached in different ways and using different mathematical tools. And the final conclusion that there are only a finite number is by no means evident *a priori*. Using the DVD *Symmetry – the dynamical way* extensively we will look at examples of *stamps* with different shapes and properties, which produce figures with different types of symmetry.

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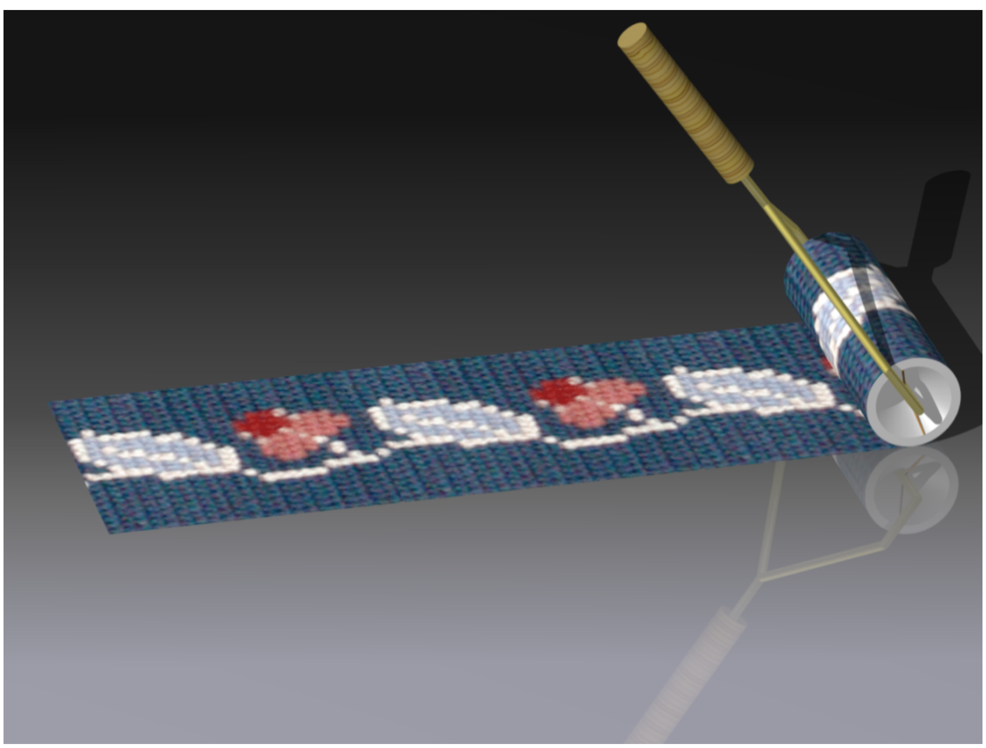
Introduction

Let us consider a paint roller with a drawing and let us roll it on a table, as in the following picture[1].



[1] Most of the pictures inserted in this paper are taken from the DVD *Symmetry – the dynamical way* (in 6 languages) produced by Atractor [1].

Figure 1: *A frieze produced by a paint roller.*

Each turn of the cylinder stamps a motif on the table and imagining that the cylinder rolls indefinitely in both directions, what we get has the following property: a translation by a vector "connecting" two equivalent points of the flowers sends the whole image onto itself and it will be impossible to distinguish between the initial and the final appearance. Translations defined by such vectors are symmetries of the object (a *frieze*) represented by that picture.

If we look carefully at this example, we see that if we replace the motif on the roll with another (asymmetrical) one, we get a different frieze with exactly the same symmetries: the final symmetries depend only on the stamp and the way the frieze is stamped, not on the initial chosen motif.

We arrive at a similar conclusion if we consider a conical *stamp* with an adequate angle, as in the following pictures for a rose with only rotational symmetries:

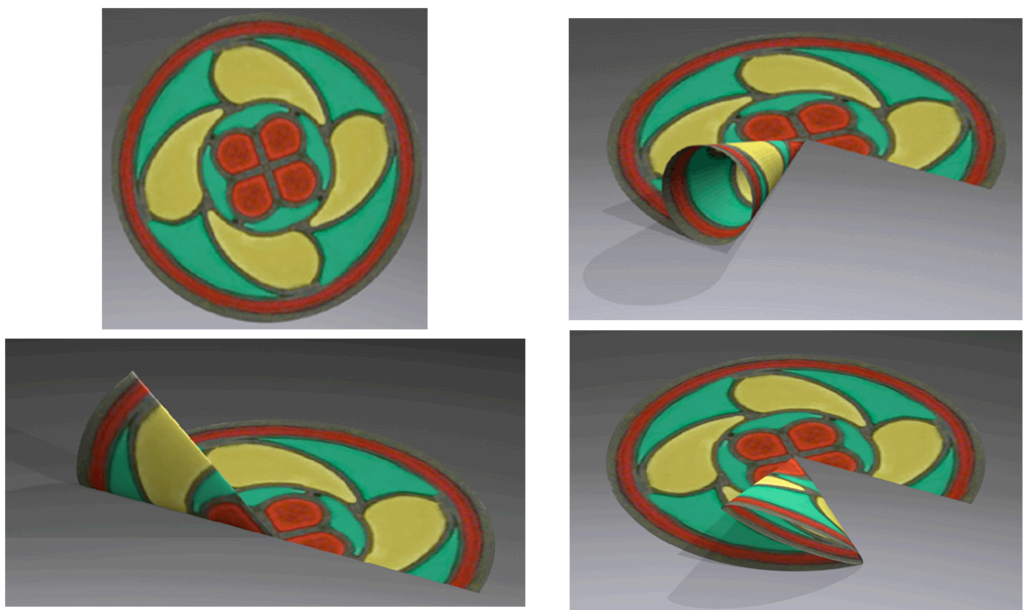


Figure 2: *A rose stamped by a cone or by a triangular-shaped flattened cone*

To produce an image with reflection symmetries, it will be enough to use some kind of *stamp* rotating (in space) around a line, as in the production of the rose of the following picture, which, unlike the previous one, has symmetries of two kinds: rotational and reflection symmetries.

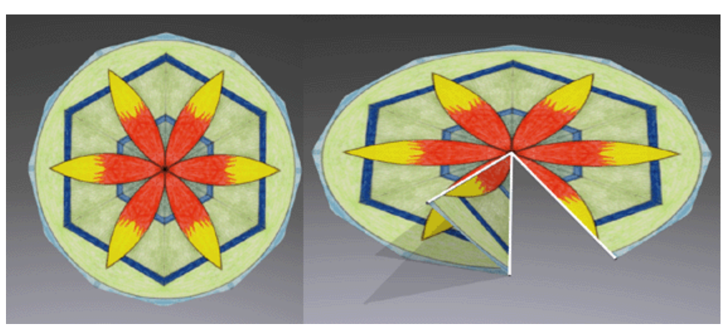


Figure 3: *Rose with 6 reflection symmetries (plus 6 rotational symmetries), together with the good stamp for it*

In this example, the stamp is a triangle with a motif on it (the same image "on both sides"). The situation was different in the case of the triangular pad of picture 2, where both sides of the pad had completely independent images.

In order to have stamps for producing all elementary types of symmetries, there is just one type missing: the *glide-reflections*, like in the frieze marked by bare feet walking on wet sand.

Figure 4: *A frieze with glide-reflections*

It is perhaps not obvious how to choose the right "stamp" for producing this frieze. Let us see with a bit more detail how to proceed, because the method we will follow here can be used in more general situations. A *minimal* region from which the whole frieze can be constructed by applying symmetries to it can be defined by two vertical lines such as indicated in the following picture

Figure 5: *A minimal region*

and a *minimal* glide reflection sending the frieze onto itself can be obtained by composing the reflection (shown on the following picture) on the middle line of the frieze with a small translation, none of these two transformations being a symmetry of that frieze.

Figure 6: *This reflection is not a symmetry; you get a symmetry by composing it with a small translation*

That glide-reflection sends the left vertical segment of Figure 5 onto the right one, in the way suggested by the colors used. If we now glue equivalent points of both segments, we get a Möbius band, which is the "good" stamp for that frieze:

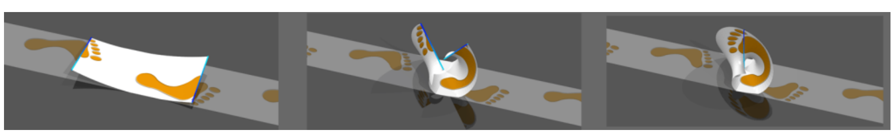


Figure 7: *Building the good stamp – the Möbius band*

Figure 8: *Testing the new stamp*

Up to now, we gave examples of four types of elementary stamps, each one good for producing pictures with one of the four types of symmetry in the plane. The idea now is to carry this correspondence between types of symmetry and stamps a bit further.

Stamping Patterns

Consider the problem of identifying how many wallpaper (doubly periodic) patterns exist on the plane, which are different from the point of view of symmetry. We shall try to give just a glimpse of the way this problem can be handled using the beautiful geometrical ideas of W. Thurston's *orbifolds*.

An *orbifold* corresponds to what we have been calling a stamp. And the idea is to establish a correspondence between patterns and stamps, such that two patterns have different type of symmetry if and only if the corresponding stamps are "different". Once this correspondence is established, counting different types of symmetry is reduced to counting (different) stamps…

Concerning the notion of orbifold and without entering into technical details, let us just say that in the same way that a surface (or 2-dimensional *manifold*) takes a disc (or a half-disc) on the plane as a local model for its points, a (2-dimensional) *orbifold* has a richer structure and we have to consider local models of different types, as suggested in the following pictures:

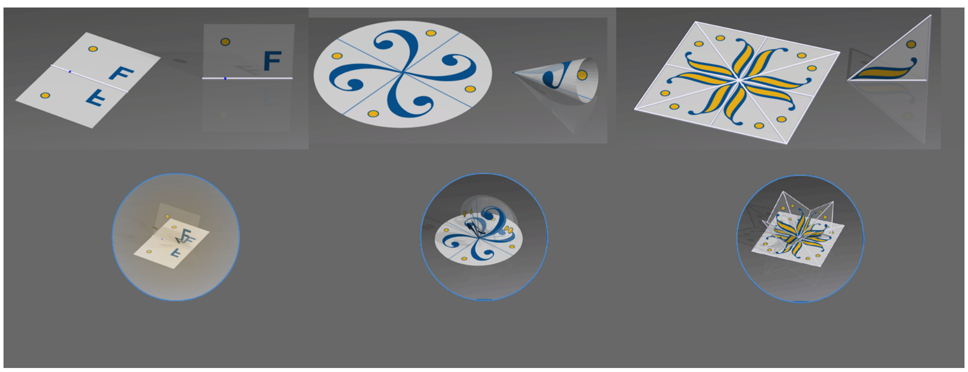


Figure 9: *Local models for points of an orbifold*

From the left, we can see a model for: a point on the border of a stamp (which would stamp a *mirror* or reflecting line on the pattern), a cone point (which would stamp a rotation center or *gyration* center of the pattern) and a *corner* on the intersection of two lines on the border of the stamp (which would stamp a rotation center at the intersection of two reflection lines).

In what follows we will take some examples of patterns and we will try to associate a stamp to each of them, by proceeding in the same way as we did previously (see pictures 7 and 8). During the talk animations will show the way the stamps stamp the pattern on the plane.

Finding Stamps for Patterns

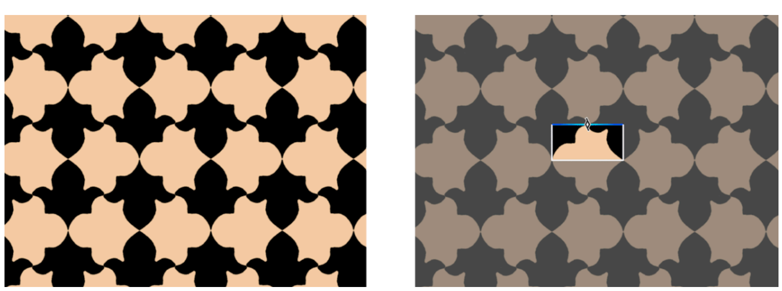


Figure 10: *(left) A pattern from Alhambra with reflections and 2-gyration centers; (right) a minimal region*

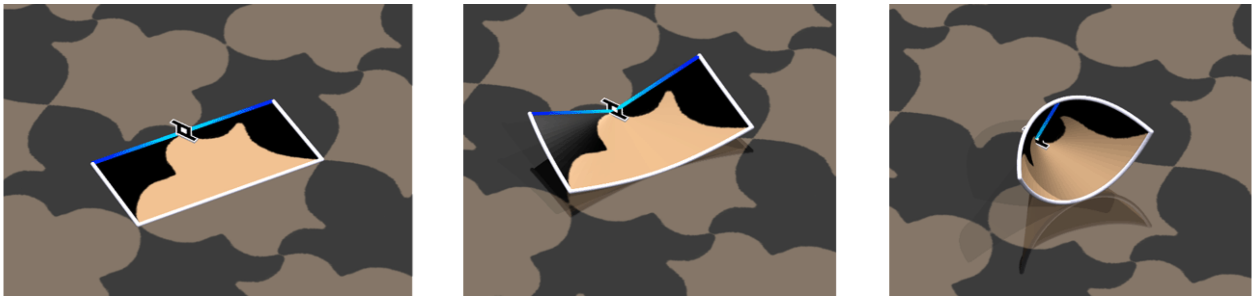


Figure 11: *Gluing equivalent points to create the stamp with a cone point and two corners (on the border)*

Figure 12: *Stamping the pattern − different steps*

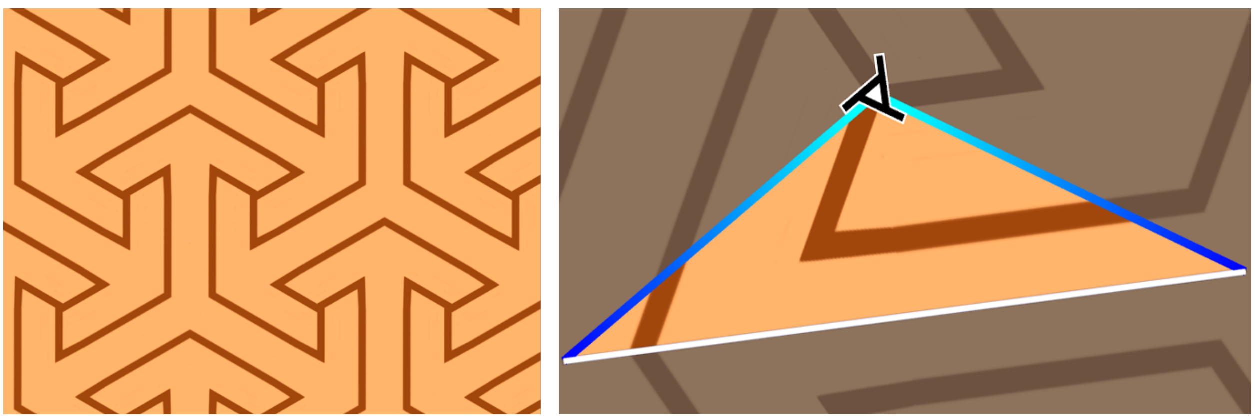


Figure 13: *Another pattern from Alhambra (left) and one of its minimal regions (right)*

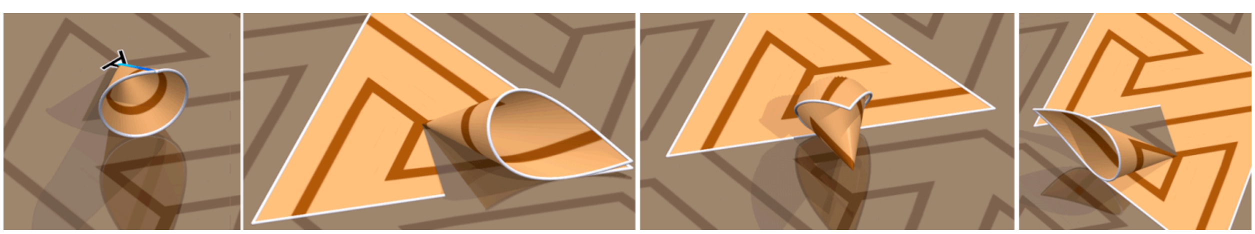


Figure 14: *The stamp (one cone point and one corner) and three steps of the stamp stamping the pattern*

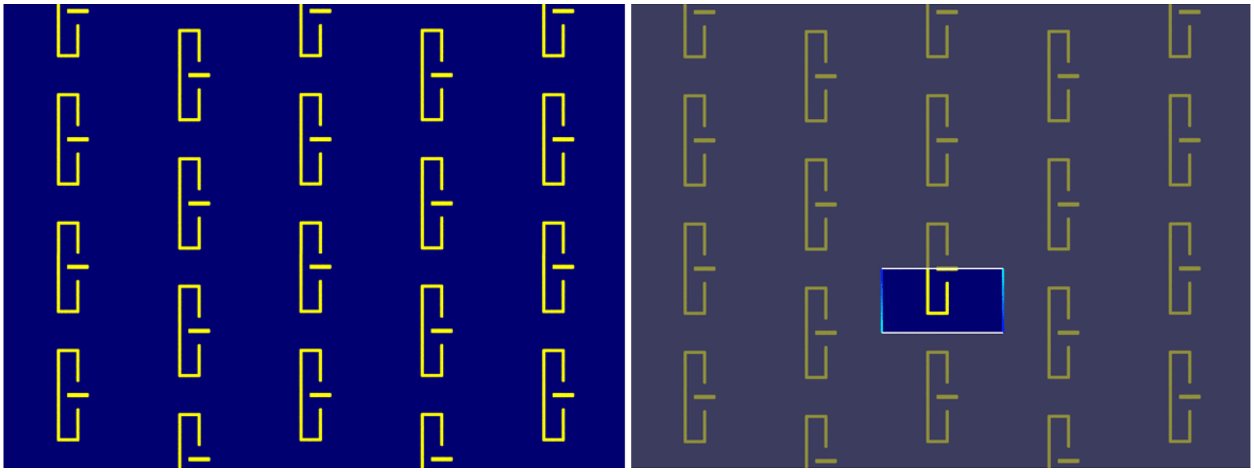


Figure 15: *A pattern (pre-Columbian America) and a minimal region*

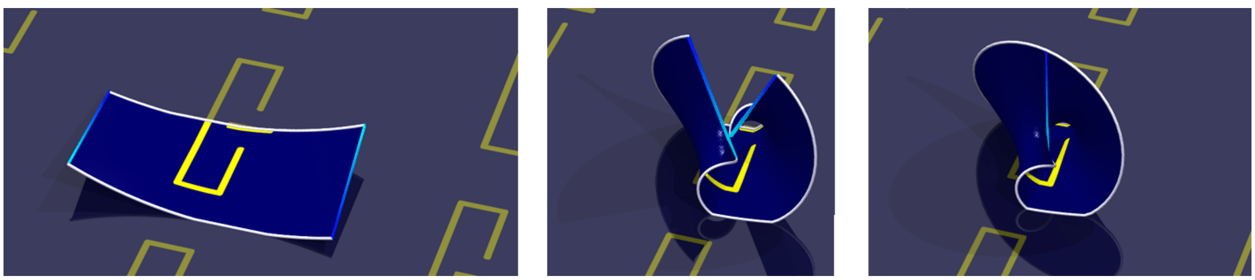


Figure 16: *Gluing equivalent points on the border to get a stamp (a Möbius band)*

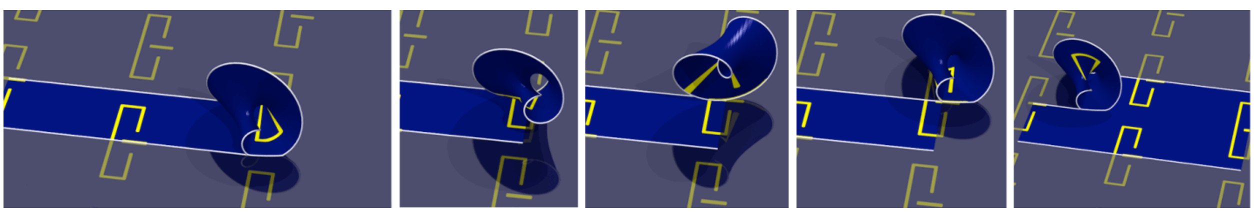


Figure 17: *The Möbius band stamping the pattern*

After finding some examples of stamps in this way, the natural idea is to try to reverse the process: if we choose a surface (possibly) with some hole(s), some cone points and some corners in an arbitrary way, will this object stamp a doubly periodic pattern on the plane? Before answering the general question, let us give an example where this happens.

A Good Stamp

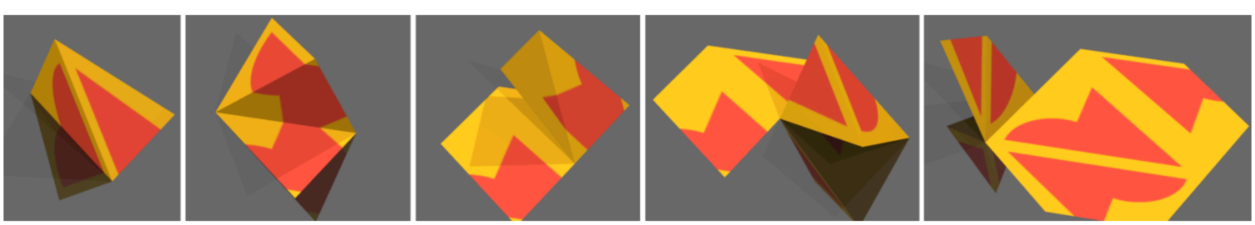


Figure 18: *A triangular pyramid stamping a pattern on the plane*

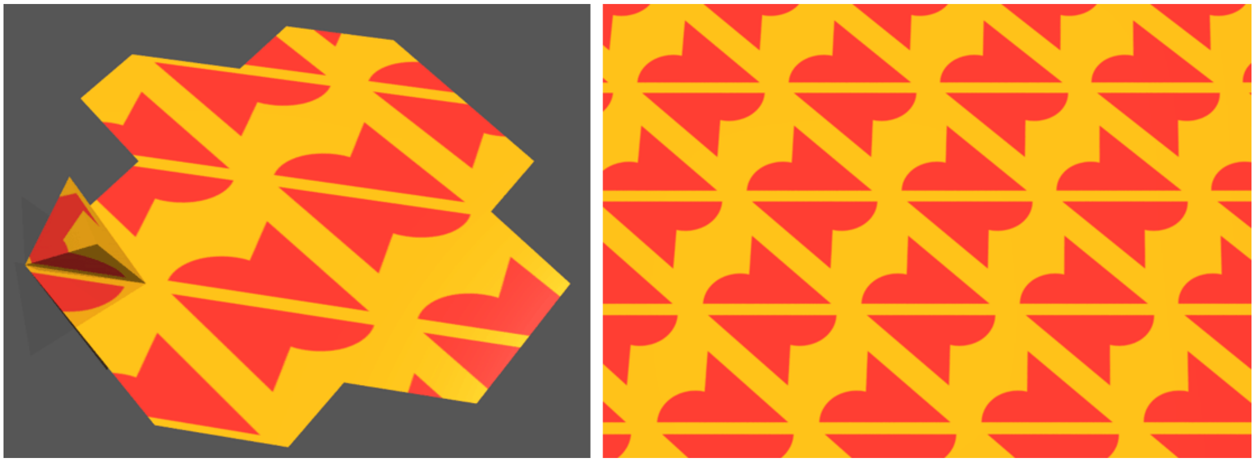


Figure 19: *Stamping further (left) and a fragment (right) of the final pattern with order 2 gyration points*

A Bad Stamp

This was an example of good stamp, but there are also "bad" stamps; here is an example:

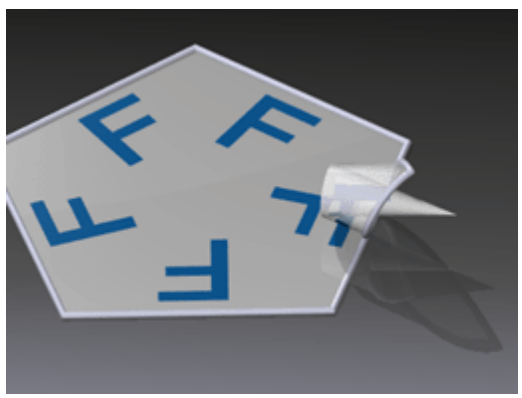


Figure 20: *Example of a "bad" stamp; it starts correctly, but the problem appears a bit later (see next figure)*

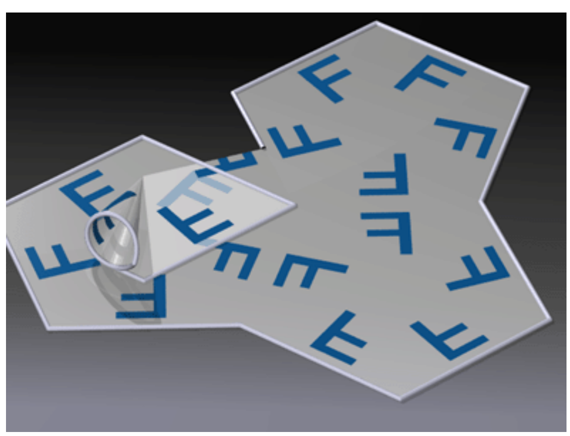


Figure 21: *Bad stamping*

Finding All Good Stamps

A natural question arises at this point: how is it possible to identify the good stamps – those which stamp the plane in a nice way, producing a pattern? A number is associated to each stamp, called its *Euler number* and the answer to that question is: the *good* stamps are those with 0 Euler number. An application in [1] allows the user to build stamps with arbitrary choices of surfaces, holes, cone points and corners and calculates the corresponding Euler number. The user should try to find the combinations which give 0 as the final result: exactly 17 for double periodic patterns and 7 for friezes.

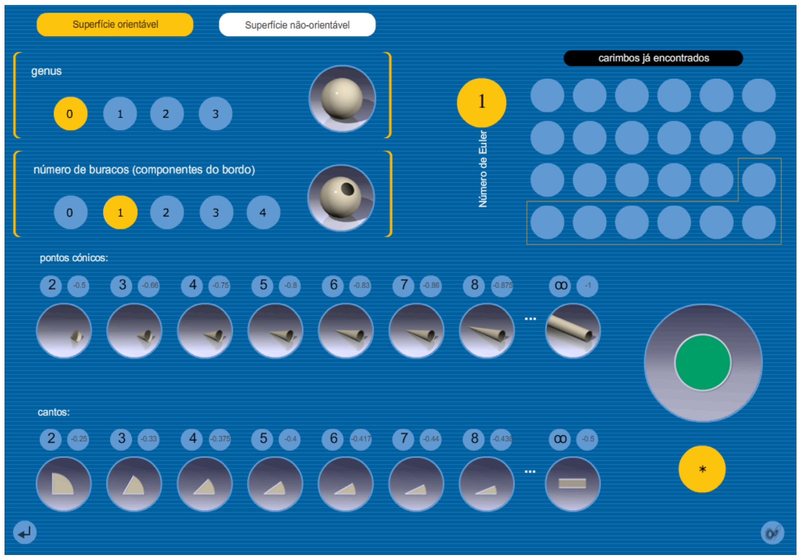


Figure 22: *Finding all "good" stamps*

And finally, here is a "photo" of all good stamps (24):

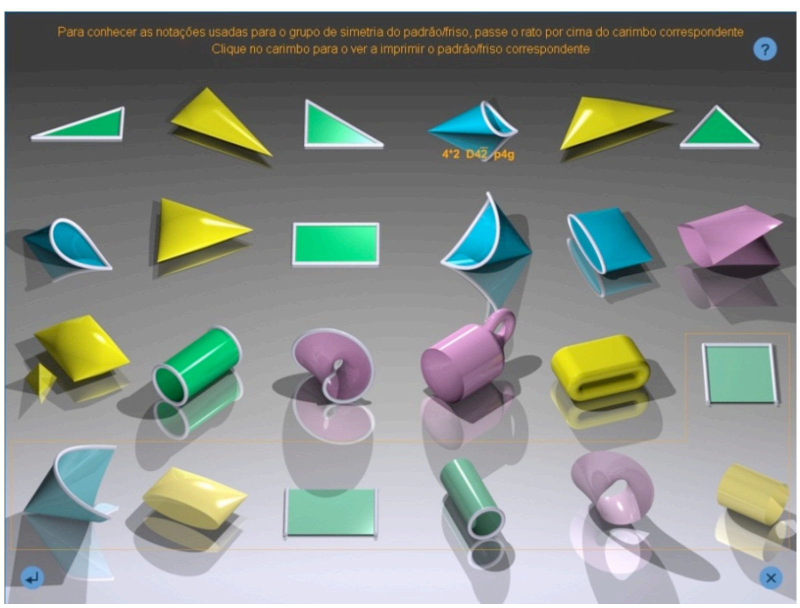


Figure 23: *All 24 stamps*

[1] Atractor, *Symmetry – the dynamical way, DVD* (http://www.atractor.pt), 2009.
[2] J. H. Conway, H. Burgiel, C. Goodman-Strauss, *The Symmetries of Things*, A.K. Peters, 2008