Creating Two and Three Dimensional Fractals from the Nets of the Platonic Solids

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Abstract

Regular polygons can be used to construct a number of solids. A large number of examples can be found at [3]. The pictures and models described in this paper start from the nets of solids that are constructed from regular polygons. The introduction considers, generally, a range of solid types that were used to create fractals in two and three dimensions. The resulting fractals have been used as a framework for displaying pictures. The body of this paper is limited to a discussion of the Platonic Solids. Whilst much progress has been made with other regular solids they will need to be the subject of a separate paper.

1. Introduction

There is a set of convex solids that are constructed from regular polygons. The Platonic Solids[2][3][4] are convex figures made up of one type of regular polygon. Archimedean solids[[2][4][5] are convex figures that can be made up of two or more types of regular polygons. Some of the Platonic solids have been known for a very long time. Three of them were probably known to the pythagoreans around 500BCE. Theaetetus, a contemporary of Plato, knew of all five Platonic solids and possibly proved that there could be only five[6]. This would have been around 360 BCE.

2. The Basic Process

In general my aim was to create the net, or at least half of a net, of a polyhedron from a polygon. The resulting net would contain smaller versions of the same polygons from which further nets could be generated. This process could then be repeated forever to produce a fractal. It was hoped that the fractalised net could then be folded to create a fractal in three dimensions. To date, I have managed to create such schemes for many of the Platonic Solids, the Archimedian solids, prisms and antiprisms.

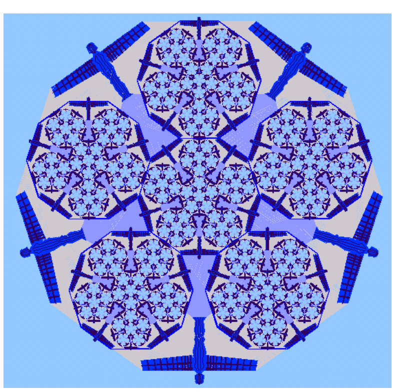


Figure 1: *Examples of two and three dimensional fractals based upon the net of a truncated dodecahedron. The picture depicted is the "Angel of the North": a very large sculpture by Gormley constructed at the head of a coal mine in Gateshead UK.*

Reference [4] is a good starting point for a few hundred models freely available as a .pdf document, although not necessarily laid out as required in this paper. Figure 1 shows examples of two and three dimensional fractals based upon the net of a truncated dodecahedron. The picture featured is based upon Antony Gormley's massive steel sculpture known as the "Angel of the North" constructed at the head of a coal mine at Gateshead in the North of England. The remainder of this paper will feature the nets of the Platonic Solids with examples of the resulting fractals in two and three dimensions.

3. A Brief Description of the Software

The software was designed to develop pictures up to 10 layers each of up to 120 megapixels and 256 colours. This being the limit of the hardware available. This allowed for A0 size pictures to be produced of a good quality. The length of these calculations necessitated regular dump and restart facilities in the software. The time involved in the creation of a high resolution picture can be considerable, frequently several hours, so the software allowed the development of a low resolution picture prior to the final version. Frames of an animated sequence can be saved so that the dynamics of the process can be studied. I allowed for polygons down to 10 pixels. I encountered no problems with overflow errors as long as I kept to polygons with edge greater than 10 pixels which, in the context of 120 megapixel pictures, was no problem. Optionally, I could include pictures into the fractal. This made provision for additional pictures to supplement a purely geometric design. Most of the pictures included were sketches of my own creation.

4. A Scheme for the Cube

A cube is probably the most readily recognised of the Platonic Solids. There are several nets that can be used to make a model of a cube. I chose one that starts with a square. Divide it into four equal squares and cut away one of them. Two such nets can be folded to create a cube. This can be seen in figure 2.

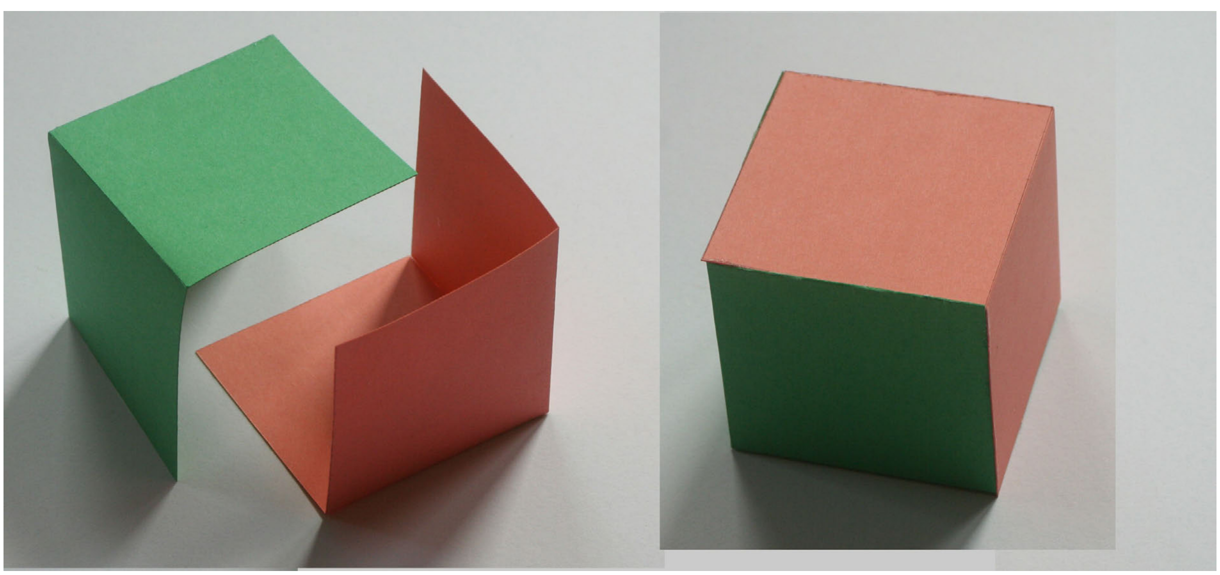


Figure 2: *The formation of a cube from two cutaway squares.*

The green square in figure 3 is divided into four, one square is removed. The same is done to each of the three smaller green squares giving nine smaller yellow squares. Each of the yellow squares was further divided giving the red squares. This process can be repeated forever. The first four generations of this process can be seen in figure 3. It was found that the resulting designs made two dimensional fractals that could be used for pieces of artwork. Additionally, they could be folded to create three dimensional fractals.

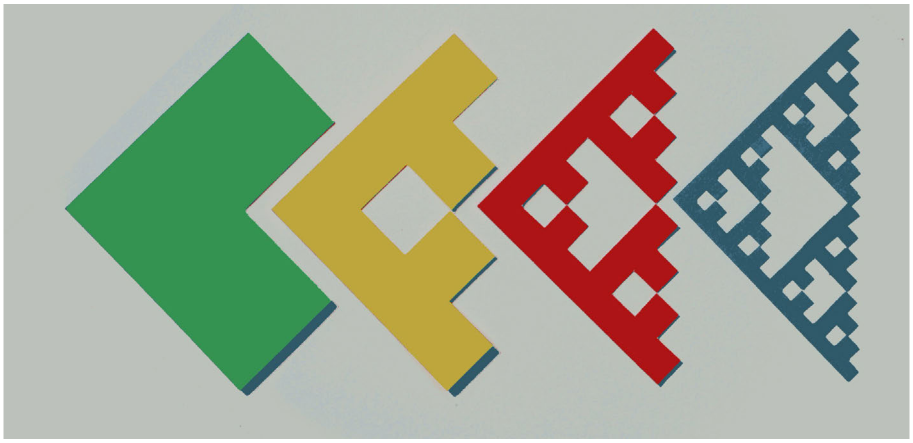


Figure 3: *The green square is divided into four, one square is removed. The same is done to each of the three smaller green squares giving nine smaller yellow squares. This can be repeated forever. The first four stages are shown here.*

Each of the nets can be folded to create fractalised versions of the original. These can be seen in figure 4. Provided the correct symmetry is chosen two of the same coloured solids can be used to create a stack of cubes. Alternatively, I created a three dimensional model, each generation of fractal fitting inside the previous one, see figure 5. Figure 6 shows a two dimensional picture called "Purple Haze". The geometry of the square has been distorted to destroy the regular symmetry. There is, nevertheless, a more subtle topological symmetry which defines the centre of a square as the place where the diagonals cross rather than half way along a diagonal.

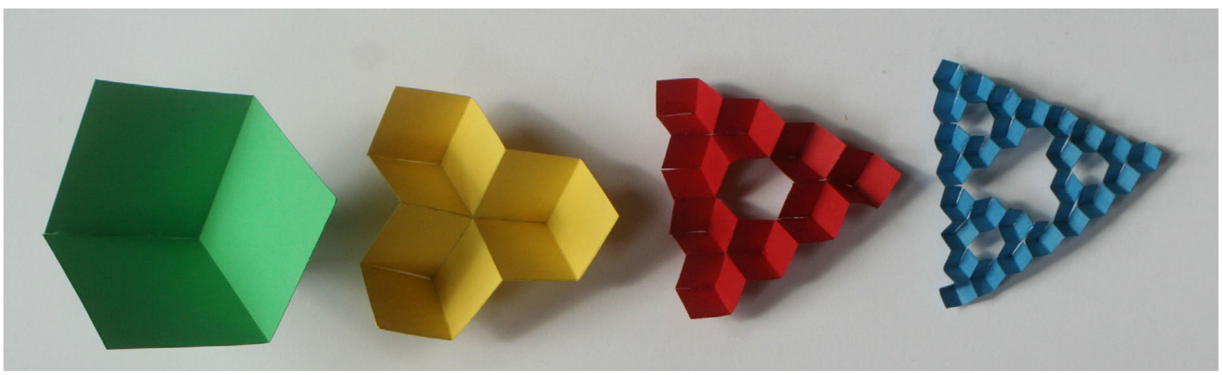


Figure 4: *Three dimensional stacks created from the nets in figure 3.*

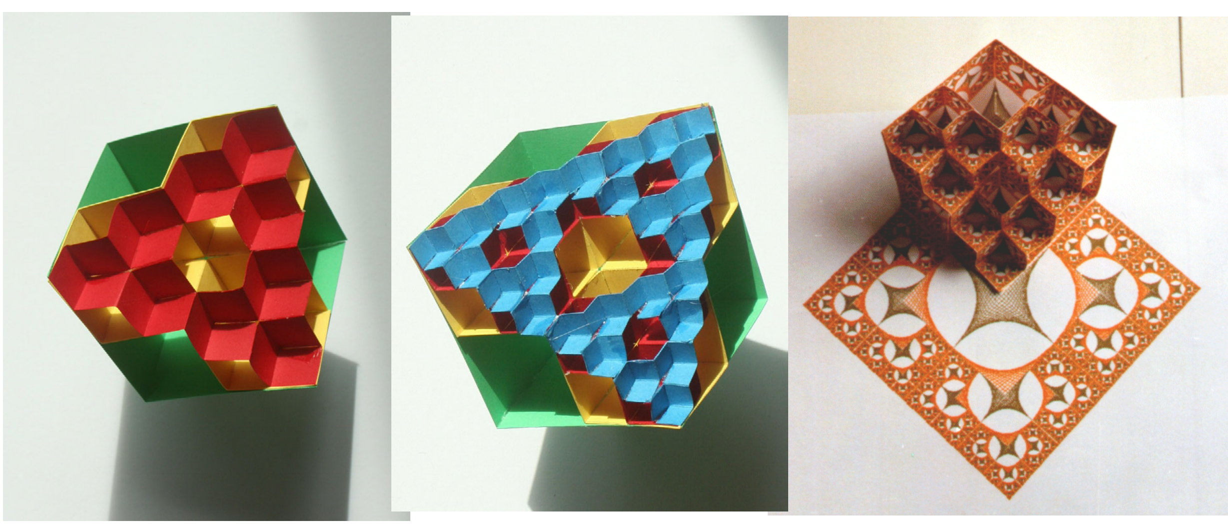


Figure 5: *Several generations of three dimensional stacks.*

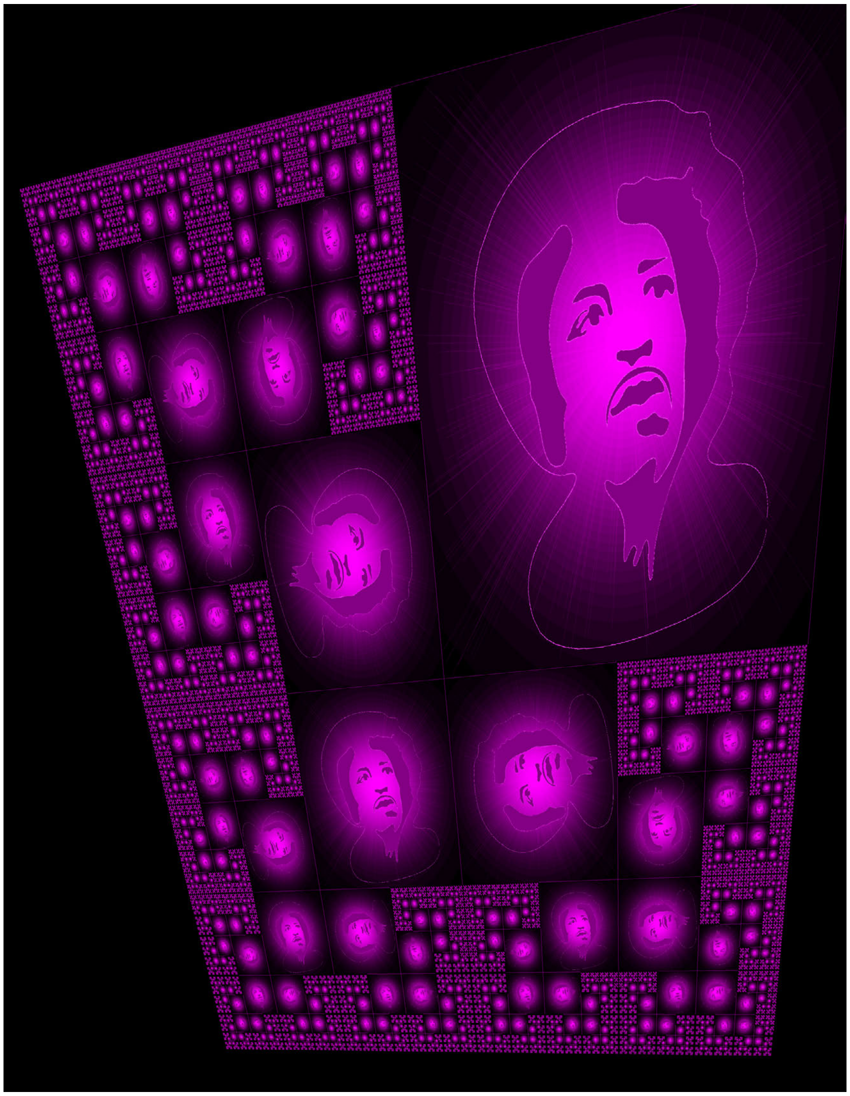


Figure 6: *This two dimensional design is called "Purple Haze" for reasons that fans of Jimi Hendrix will understand. The geometry of the square has been distorted to destroy the regular symmetry. There is, nevertheless, a more subtle topological symmetry which defines the centre of a square as the place where the diagonals cross rather than half way along a diagonal.*

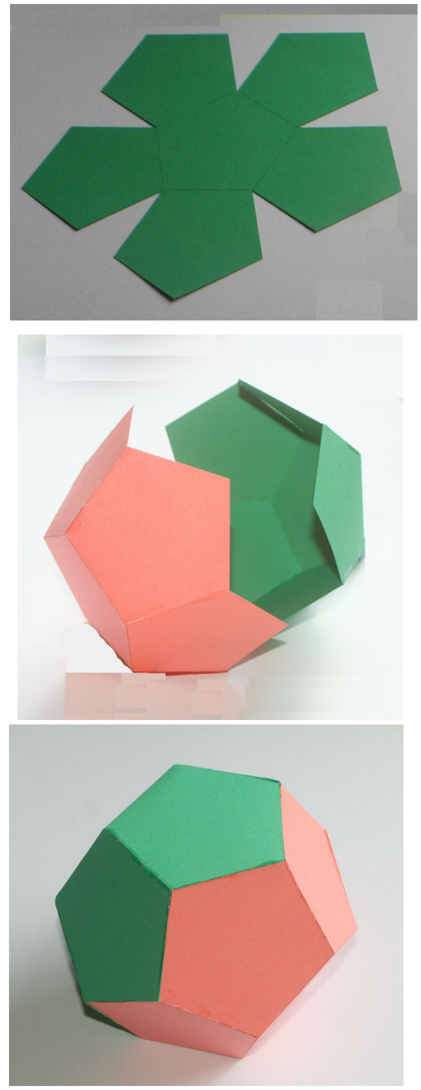


Figure 7: ***T****he net for a dodecahedron is based upon a pentagon.*

5. A Scheme for the Dodecahedron

I chose the net for a dodecahedron that starts as a pentagon. This was divided into six further pentagons. Two such nets can then be folded and made into a dodecahedron. This can be seen in figure 7. Each of the six pentagons can then be divided into a further six pentagons. This process can be repeated to create even smaller pentagons. The first three stages of this process are shown in figure 8. Figure 9 illustrates how the nets can be folded to create a three dimensional model. Figure 10 is the design of a Christmas Card that used a sketch of a "Rocking Santa" to fill the pentagons. Again the pentagons were distorted; the corners of the pentagons found from the intersection of diagonals rather than measurement.

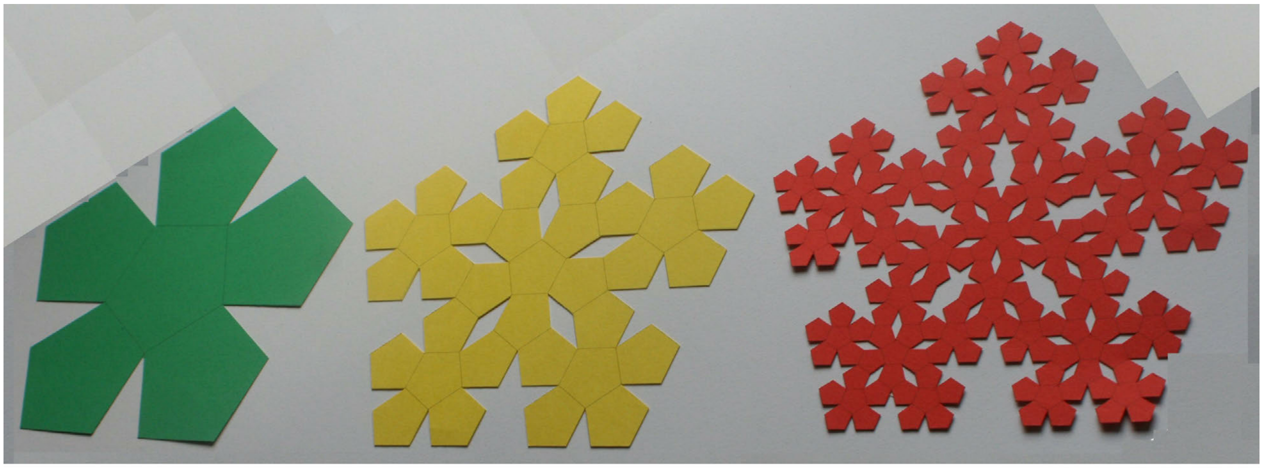


Figure 8: *The green pentagon is divided into 36 pentagons shown as yellow in the picture Each yellow pentagone can be further divided into six pentagons. The resulting 216 pentagons are shown as red. This can be repeated for ever. The first three stages are shown here.*

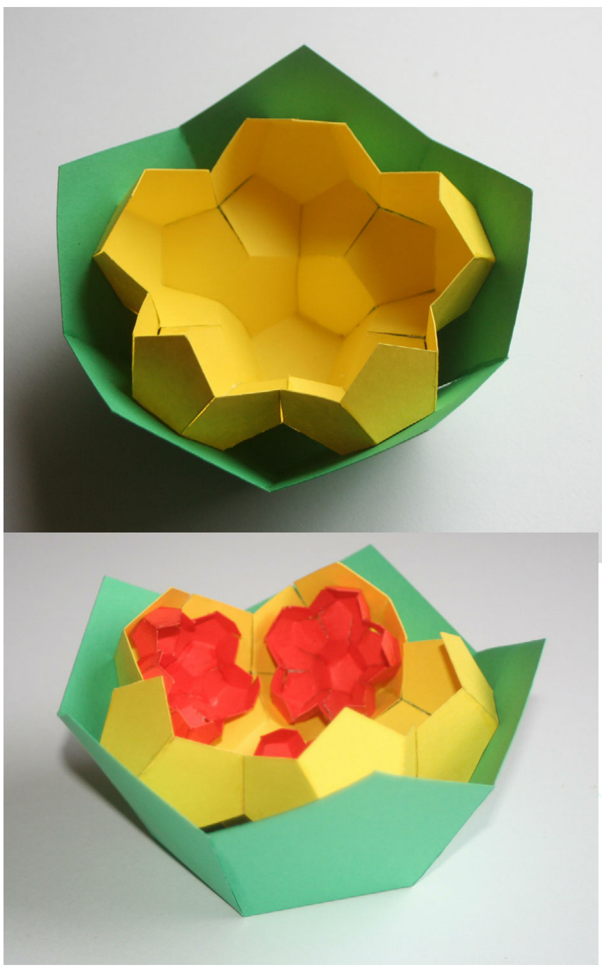


Figure 9: *The nets can be folded to create three dimensional objects.*

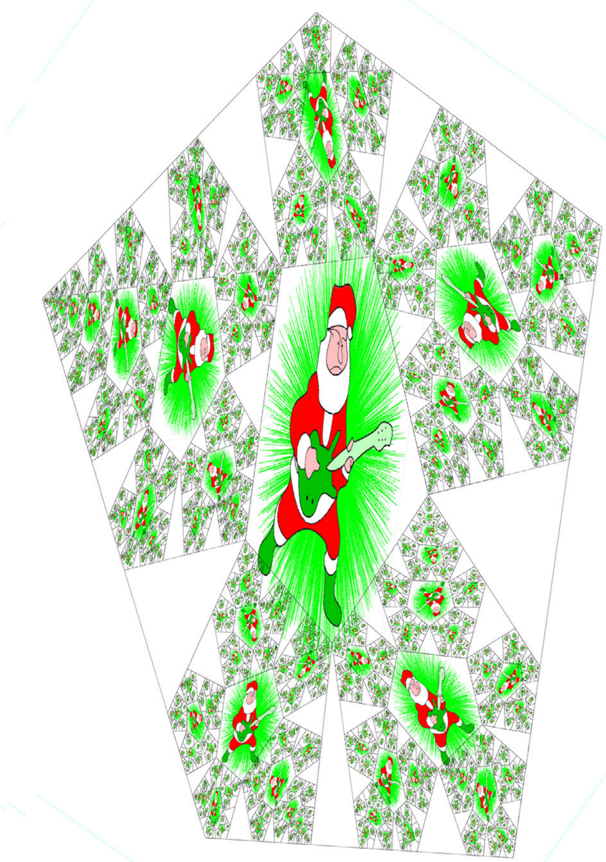


Figure 10: *This two dimensional fractal uses a sketch of a "Rocking Santa" to create a two dimensional fractal which I used on a Christmas card.*

6. Schemes using the equilateral triangle.

There is one and only one Platonic solid using the square and one using the pentagon. This is because only three of the polygons can be fit around a point to form a vertex, The remaining three Platonic solids are based upon the equilateral triangle. To form a vertex at least three triangles are required. Six will result in a tiling which fills the plane and not leave room for a cut out needed to fold up a vertex. Three triangles corresponds to the tetrahedron, four the octahedron and five the icosahedron., see figure 11

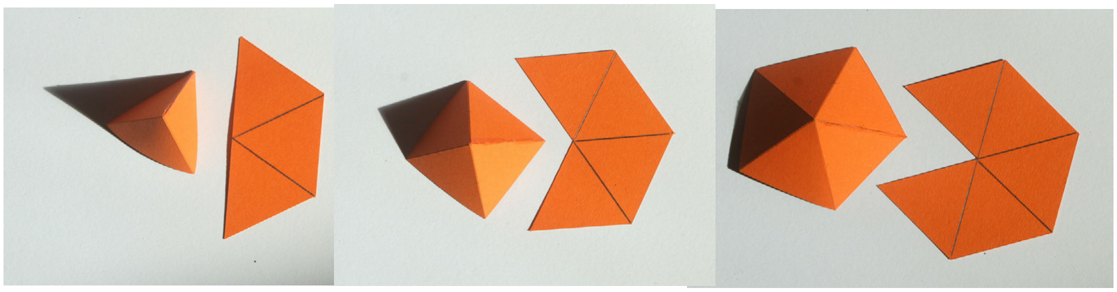
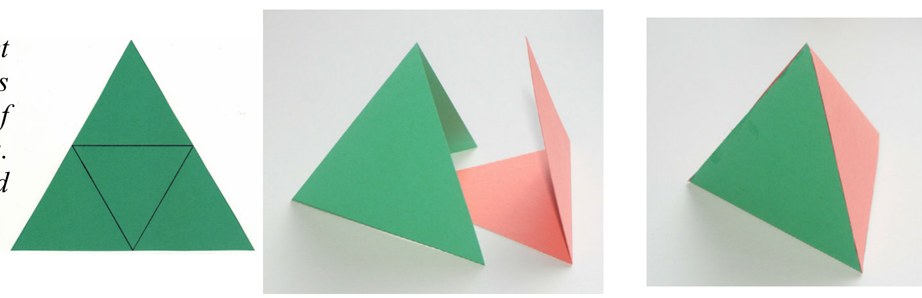


Figure 11: *The vertices for the tetrahedron, Octahedron and Icosahedron are formed from equilateral triangles.*

7. A Scheme for the tetrahedron

The tetrahedron can be created from a single equilateral triangle divided into four smaller triangles. To continue with the idea of using two identical half nets, however, two of the small triangles need to be cut away. Two such half nets can be used to make a tetrahedron, as shown in figure 12. This process can be repeated, as before, to fractalise the net. Figure 13 shows a three dimensional model based upon this process. Figure 14 shows a two dimensional design using my sketch of "Buddy Holly" in holly colours.

Figure 12: *The half net for the Tetrahedron is formed from a pair of equilateral triangles. Two such pair are used to make a tetrahedron.*



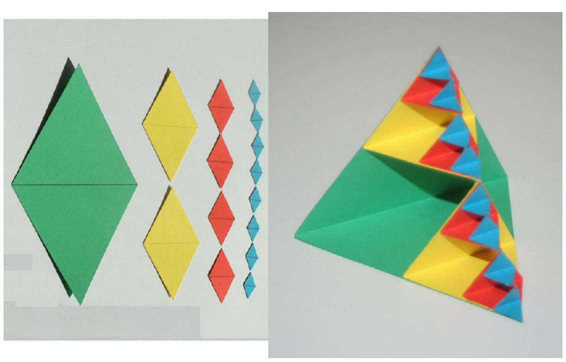
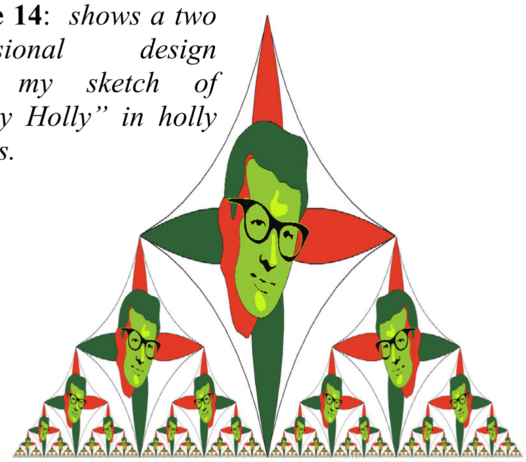


Figure 13: *The development of the net to create a three dimensional model.*

Figure 14: *shows a two dimensional design using my sketch of "Buddy Holly" in holly colours.*



8. A Scheme for the Octahedron

The Octahedron can be created from a single equilateral triangle divided into four smaller triangles. This can be folded up to create a half net, however, nothing needs to be cut away. Two such half nets can be used to make a Octahedron, as shown in figure 15. Each triangle can then be subdivided into four triangles. This process can be repeated, as before, to fractalise the net, see figure 16. Figure 17 shows a three dimensional models based upon this process using the first three generations. I was struck by the particularly rigid structure that could be formed and will make a point of looking into this aspect further. Figure 18 shows a two dimensional design using my sketch of an Alligator entitled “See You Later”. It is interesting that the octahedron needs nothing cut away to create the net. To create enough space for a picture I used the central triangle for the picture and fractalised the three others I could have used a different triangle to create a different design. The resulting design resembled the Sierpinski Sieve[1]. As with the earlier designs I have distorted the symmetry.

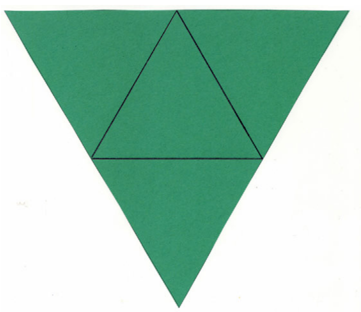
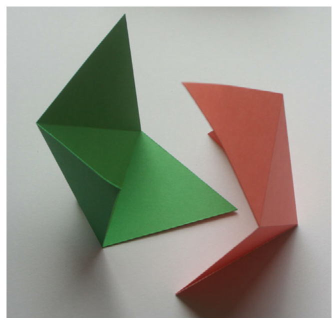
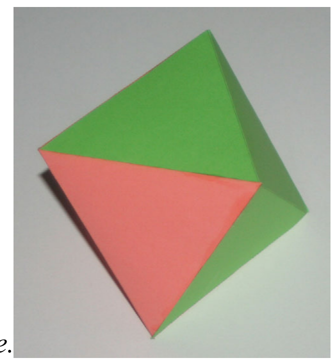




Figure 15: *The net for an Octahedron is based upon the equilateral.triangle.*

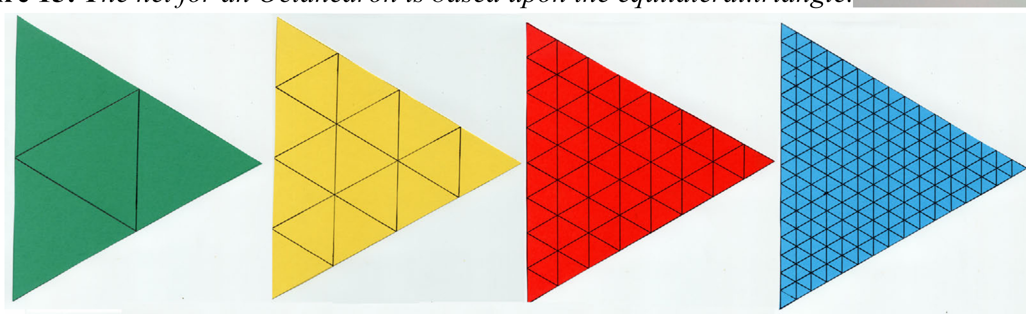
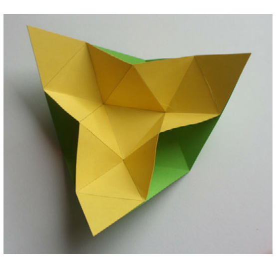


Figure 16: *The first four generations of half net for the Octahedron. No cut out is required to form the net.*



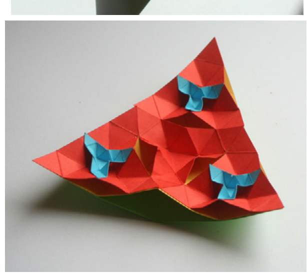


Figure 17: *The models created from the first four generations of net. Some cutting is required to form the models but nothing is removed.*

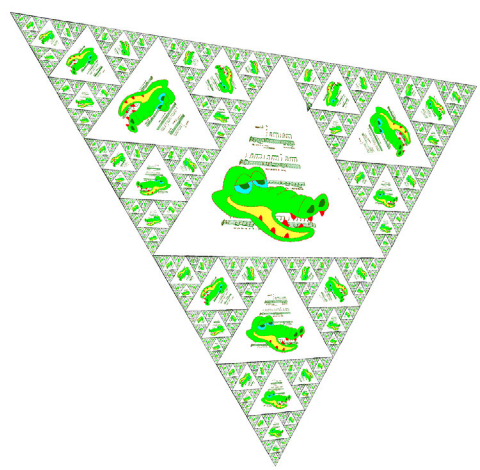


Figure 18: *This design is similar to the Seirpinski Sieve although it is possible to create other designs. Note the distortion of the symmetry as with earlier designs.*

9. A Scheme for the Icosahedron

The Icosahedron can be created from a single equilateral triangle divided into sixteen smaller triangles. Six of them are cut away to form the ten triangles of the half net (see figure 19). This can be folded up to create a half of an icosahedron. Two such half nets can be used to make a complete Icosahedron, as shown in figure 20. Each triangle can then be subdivided into a further ten triangles. This process can be repeated, as before, to fractalise the net, see figure 19. Figure 20 shows a three dimensional models based upon this process using the first two generations. Figure 21 shows a two dimensional design using my sketch of an Evis Presley entitled "Sun of Memphis". I was able to utilise the cut away space for images, leaving all the triangles for the next generation.

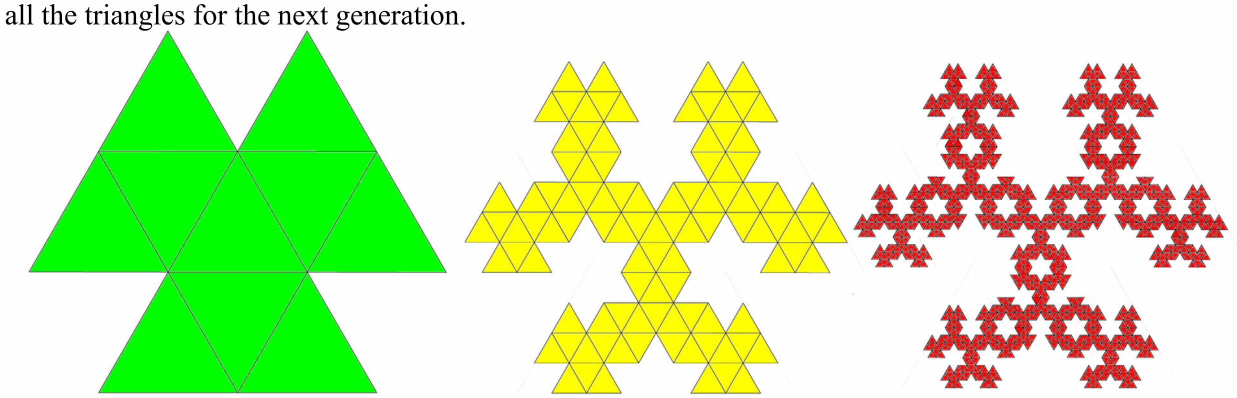


Figure 19: *The first three generations of half net for the Icosahedron.*

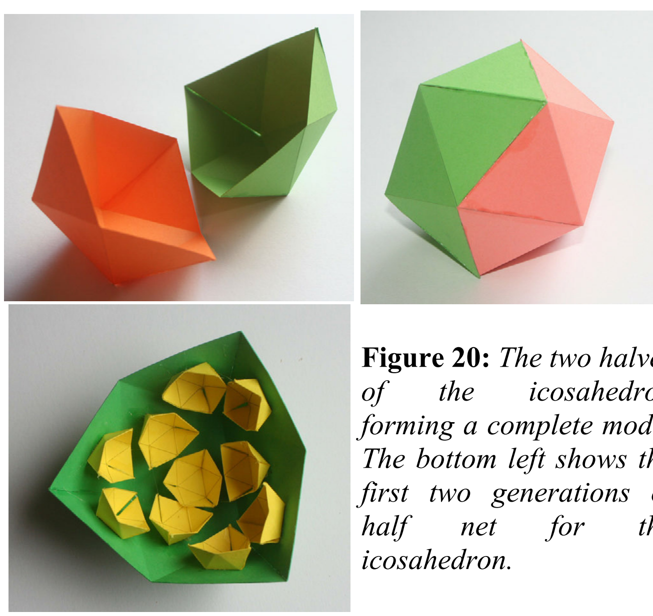


Figure 20: *The two halves of the icosahedron forming a complete model The bottom left shows the first two generations of half net for the icosahedron.*

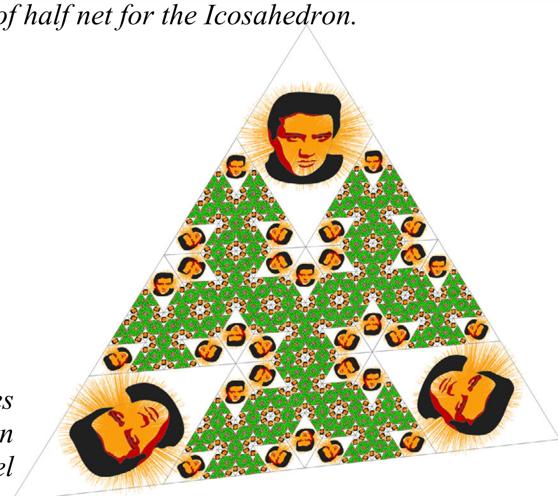


Figure 21: *This shows a two dimensional design using my sketch of "Elvis Presley"; the design is entitled "Sun of Memphis".*

References

[1] Hans Lauwerier, *Fractals Images of Chaos,* ISBN 0-14-014411,-0 Penguin, 1991
[2] David Wells, *Curious and Interesting Geometry,* ISBN 0-14-011813-6, Penguin, 1991
[3] http://en.wikipedia.org/wiki/Platonic_solid(accessed 12.12.2010)
[4] http://www.korthalsaltes.com (accessed 12.12.2010)
[5] http://en.wikipedia.org/wiki/Archimedes(accessed 12.12.2010)
[6]http://www.mathpages.com/home/kmath096/kmath096.htm(accessed 12.12.2010)