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Ribbon Edges: a New Impulse for Geometric Sculpture

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Abstract

This paper describes an algorithm for decorating the edges of a regular tetrahedron with helical ribbons to generate new forms. The parameters of the algorithm, which include amount of twisting and cross section of the extruded ribbon, yield a range of interesting phenomena. The link known as the Borromean rings makes a surprising appearance. The extension of these results to the compound of five tetrahedra is described, and examples are shown. Applications to 3D printing are discussed.

Motivation. One consequence of the advent and spread of three-dimensional (3D) printing technology in the last decade has been to complement the visual mode of man-machine interaction with a tactile one. The goal of the current research is to explore geometric representations of polyhedra that are particularly appropriate for production of 3D prints. The particular figure which served as the impulse-giver for this work is the well-known compound form composed of five regular tetrahedra, whose vertices lie on the vertices of a regular dodecahedron. A simple, visually-simple representation of this figure is shown in Figure 1. In the process of creating this print the question arose whether there were alternative representations for this form which might have more interesting tactile or visual features. A sample of the work presented here is shown in the right image.

Overview. This research proceeded in two steps. First, the case of a single regular tetrahedron was explored. Then, results obtained there were specialized for the case of the 5 interlocked tetrahedra. The main idea presented here is to replace each edge of the tetrahedron with a pair of helical ribbons. At the vertices, these ribbons connect to ribbons coming from neighboring edges which share the vertex. A model was developed with a range of parameters to allow variation of the size, shape, and amount of twisting of the ribbons. The current implementation restricts to the regular tetrahedron, but we believe the basic idea can be extended to general polyhedra.

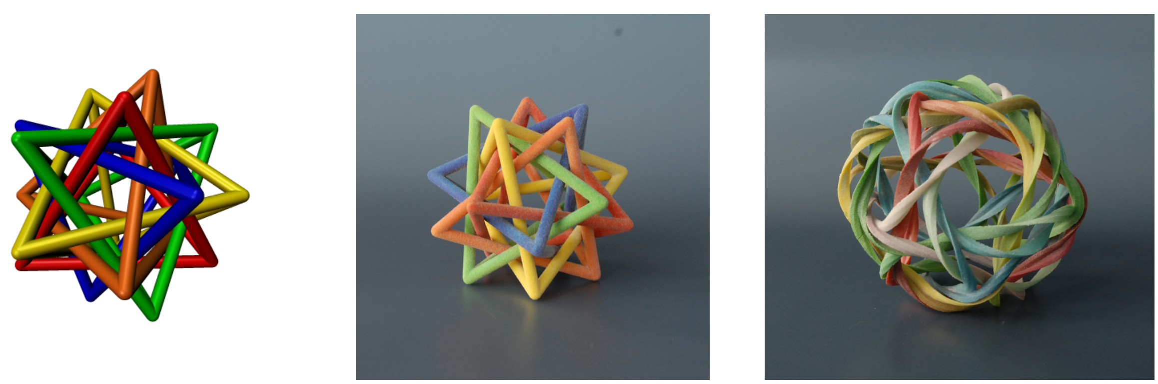

Figure 1 : *Left: Computer image of five interlocked tetrahedra. Middle: resulting 3D print. Right: 3D print based on ribbon edge representation.*

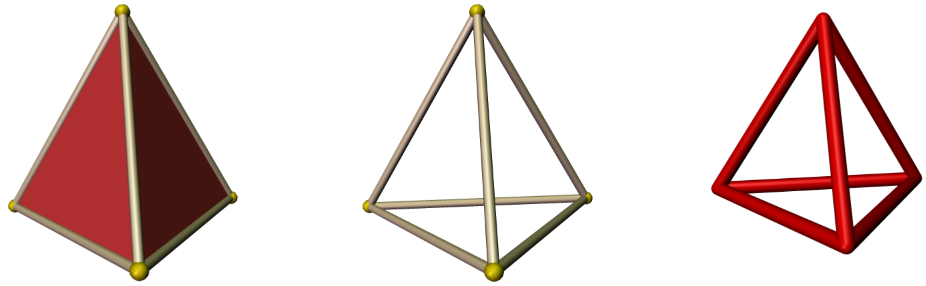

Figure 2 : *Left: Typical ball-and-stick rendering of a tetrahedron, with and without faces. Right: Stick-only model without faces.*

Edge-based polyhedral representations. An oriented polyhedron consists of a set of faces, edges, and vertices, such that every edge is shared by two faces and two vertices. A typical rendering method to display all geometric elements is the ball-and-stick method illustrated in Figure 2.

For this project the decision was made to focus on edge-based representations, and omit explicit reference to vertices and faces. To obtain a purely edge-based representation, we define a stick to be the set of all points a fixed distance from the line segment representing the edge. The resulting stick includes hemispheres at the ends of each edge. Such a model was used to create the right figure in Figure 1.

This model represents an extreme point in the space of edge-based representations. As a 3D form, it consists of two very different sorts of points: the interior of the edges, characterized by a uniformly straight cylinder; and vertices, where all incoming directions are absorbed into a sphere.The idea behind this work is to soften this strong dichotomy while remaining true to the form of the tetrahedron. The use of ribbons as described in the introduction will now be more carefully presented as a solution to this problem.

Overview of the algorithm. The solution proposed here takes it starting point from a rounded-off figure as shown in Figure 3. Such a figure is obtained from the edges of the tetrahedron by replacing the ends of each edge with circular arcs that join with C^1 continuity. Such a rounded-off form serves as a *core curve*, which is then *decorated* with a set of helical ribbons that wind around it. Each original edge is decorated with a double helix. At the rounded corners, the ribbons in the double helix split apart. Each follows a different branch of the core curve, onto another edge of the figure which meets this vertex. In this way, a closed set of ribbons are created consisting of these helical pieces. The core curve is not part of the final form.

We impose a further condition: the whole configuration has the rotational symmetry of a regular tetrahedron. The symmetry group can then be used to generate the rest of the form from a single fundamental domain. For the core curve, this reduces to a curve which begins in the middle of a tetrahedral edge and goes around a corner of the tetrahedron to the middle of an adjacent edge. We call this the *fundamental curve*, and

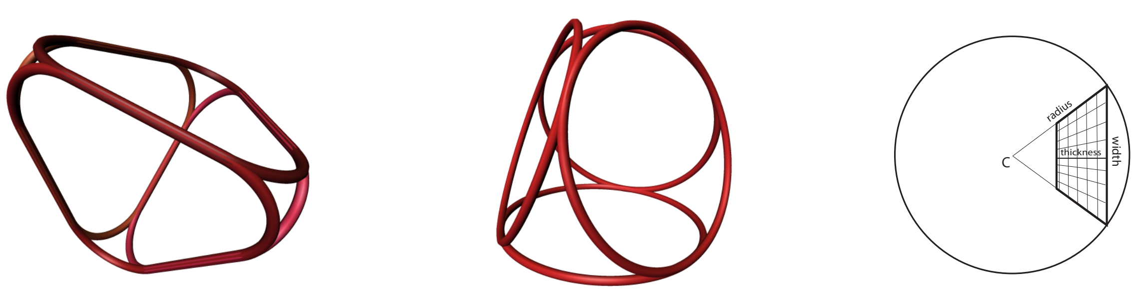

Figure 3 : *Left: Rounding off the faces. Middle: circular faces. Right: ribbon cross section.*

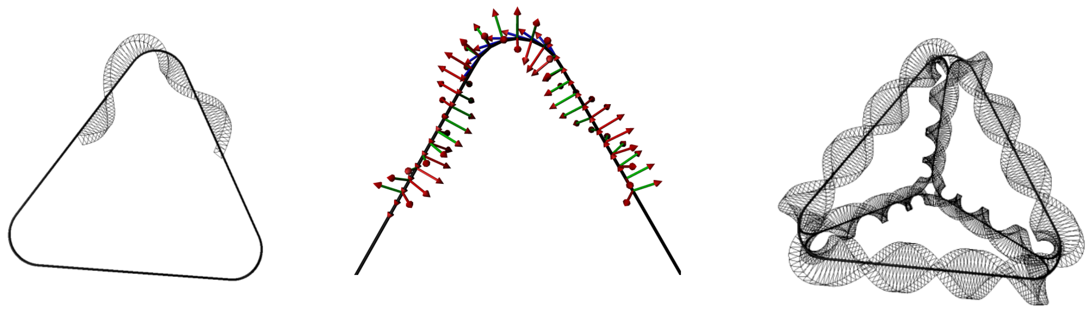

Figure 4 : *Ribbon with 4 half turns: wireframe, frame field, and the complete form.*

a ribbon decorating this curve is called a *fundamental ribbon*. See Figure 4.

Generating the ribbons. To generate the ribbons, a closed two-dimensional polyline is simultaneously rotated around and translated along the arc-length-parametrized fundamental curve, using standard techniques of differential geometry of curves ([3]). The frame field generated along the curve by this motion is shown in Figure 4. The resulting extruded surface forms the helical ribbon. The end product is characterized by a number of parameters:

half turns How much does the fundamental ribbon twist around the core? This angle is measured with respect to parallel displacement along the curve. The number of half turns must be an integer value in order to close up with the adjacent copies. See Figure 5. We consider only non-negative values since negative values yield mirror images of positive ones.

radius The ribbon is scaled to lie within a tube of this radius around the core curve. Smaller tubes accentuate the linear quality of the resulting form; larger values can lead to self-intersection of the ribbons.

phase The initial angle of rotation of the cross section with respect to the axis of the curve is chosen so that the helical partners lie exactly opposite with respect to the core curve. When the number of half turns is even, then there is an optional parameter *phase*, which increases this initial angle of rotation. Under the phase change the paired ribbons approach each other, since they rotate in opposite directions. In Figure 6 (left) the phase was chosen so the ribbons touch. For odd number of half-turns, non-zero phase leads to ribbon pieces that do not properly connect.

Note that the rounding of the core curve is not a separate parameter. In the current implementation, the amount of rounding depends on the number of half turns, so that the amount of turning along the circular arc is always one half turn. This restriction could be removed in future.

Constructing the cross-section. Consider the plane perpendicular to the fundamental curve at its beginning point. Within this plane, consider the circle of radius *radius* (from above) centered on this point. The cross section is a quadrilateral defined within this circle as follows (see Figure 3, right):

width This controls the length of a chord within the circle that forms the outer segment of the polygon. It is normalized so a value of 2 yields a diameter.

thickness This controls the radial thickness of the cross section. It is normalized so a value of 1 extends all the way to the core curve. Thickness values less than one result in ribbon pairs that do not touch. For 3D printing, one needs in this case to either adjust the other parameters to limit the amount of movement, or insert struts or other support structures to provide rigidity.

taper factor A value of 1 yields a shape as shown in Figure 3; a value of 0 yields a rectangle rather than a trapezoid; intermediate values interpolate between the two extremes.

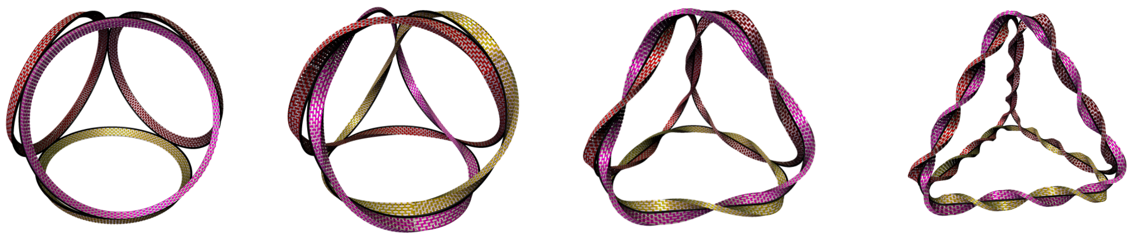

Figure 5 : *Varying the half turn parameter: 0, 1, 2, and 4 half turns.*

Finally, this polyline is rotated around an axis in its plane passing through the core curve, so that it is aligned perpendicular to the direction of the screw motion. The resulting ribbon then consists of quadrilaterals that are close to rectangles.

The final form consists of the images of the fundamental ribbon under the 12 elements of the direct tetrahedral group. For odd number of half turns, this results in 3 ribbons, each of which wraps around 4 edges, omitting one pair of opposite edges. For even number of half turns, there are four ribbons, one for each face. The ribbons are then each assigned a different color. Most of the pictures shown include a woven texture map which incorporates this color choice, and provides optimal detail level for stereo pair fusing.

Results. Figure 5 shows varying half turn values with otherwise fixed parameter values. Note that the case of 0 half turns has been implemented, although strictly speaking it does not yield helical ribbons. One surprising result is that the case of one half turn yields the famous link, the Borromean rings. We use this case to illustrate a variety of different cross section settings in Figure 6.

A printed version of the compound figure with 2 half turns is shown in Figure 1. A wide selection of stereo pair images of this compound form can be found at [2]. One surprising result was that the case of 0 half turns yields a wide range of interesting forms, consisting of 20 identical rings, arranged with dodecahedral symmetry. The case of 1 half turn yields non-intersecting ribbons only for relatively thin ribbons. Two half turns provide adequate space for a wide variety of non-intersecting ribbons.

Acknowledgements. The 3D prints were produced at the 3D Labor of the Technical University Berlin. The images were created with Java applications based on the 3D scene graph package jReality [1].

References

[1] jreality, 2006-2010. `http://www3.math.tu-berlin.de/jreality`.

[2] `http://picasaweb.google.com/cgunn3/RibbonEdgedTetrahedron`.

[3] Barrett O'Neill. *Elementary Differential Geometry*. Academic Press, 1997.

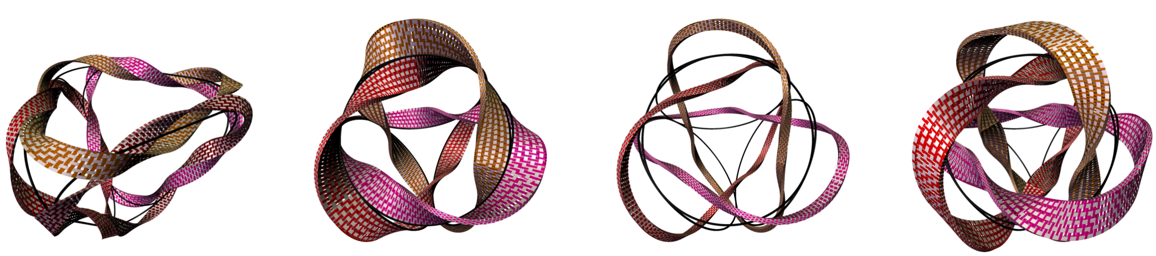

Figure 6 : *Left: non-zero phase for 2 half-turns. Right: various cross sections: thick, skinny, wide.*