Creative Learning with Giant Triangles

Simon Morgan
University of New Mexico, New Mexico Consortium
morga084@gmail.com

Jacqueline Sack
University of Houston-Downtown, Department of Urban Education
sackj@uhd.edu

Eva Knoll
Mount Saint Vincent University, Faculty of Education.
eva.knoll@msvu.ca

Abstract

Participants will review, explore and develop geometric concepts and vocabulary, through guided discovery, by building polyhedra with giant, brightly-colored, connectable equilateral triangles. Activities will be geared for teachers of all levels. Learning issues and instructional strategies will be related to teacher-student 'hidden contracts' and the Van Hiele Model of Geometric Thought.

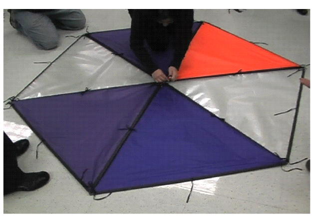

Figure 1:
Angle Sum 1

Figure 2:
Angle Sum 2

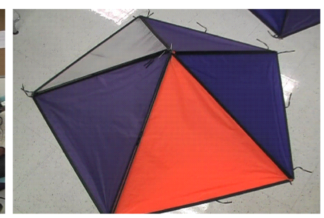

Figure 3:
Angle Sum 3

Figure 4:
Trial and Error 1

Figure 5:
Trial and Error 2

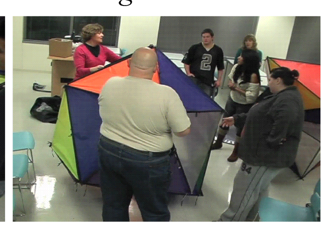

Figure 6:
Counting Faces

Figure 7:
Sitting Inside

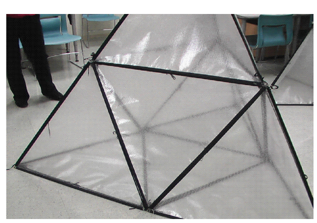

Figure 8:
Hidden Shapes

Figure 9:
Cube surprise

1 Introduction

We shall discuss developmental and instructional issues (section 2), instructional strategies (section 3) and mathematical content (section 4) of activities using the giant triangles. The term 'instructor' will be used for the person(s) supervising the workshop activities and 'students/participants' for people taking part in the activities under the supervision of the instructor.

1.1 Scope of the Workshop. Participants will experience mathematical learning through creative activity using giant, brightly-colored, connectable equilateral triangles. They will assemble them into polyhedra which can be experienced from both inside and out. An essential property of these triangles is the flexibility with which they connect. Whole shapes can deform as one part of the shape is changed (figures 4 and 5), and a wide variety of shapes can be made, unconstrained by rigid dihedral angles along edges.

The core workshop activities will include making pyramids and the Platonic solids with triangular faces: the tetrahedron, octahedron and icosahedron. These will be used to illustrate instructional and developmental issues (section 2), instructional strategies (section 3) and for in depth mathematical discussion of the topics of regularity, symmetry and counting (section 4). Further activities will depend on time and the way the workshop develops. There will be a discussion of ways of using and extending these learning activities in the classroom and the use of alternative materials for all grade levels. These include use of manipulatives, such as card or Polydrons™, which can be assembled into polyhedra with different shaped faces.

1.2 Background of Giant Triangle Activities. The photographs shown here are from course activities for candidate teachers at the University of Houston-Downtown, Department of Urban Education and from a class at an urban middle school of the Houston Independent School District, both led by the second author. An account of the materials, a series of lessons and work with low performing students is given in [2]. The activities have been developed over a 10-year period by Jacqueline Sack, Eva Knoll and Simon Morgan (see [1], [2] and [3]) and have been used for in-service teacher workshops, and mathematics methods classes.

2 Developmental and Instructional Issues

2.1 The Student-Teacher 'Hidden Contract'. The challenge of developing deep content knowledge and pedagogical content knowledge [e.g., 5, 6] for both in-service teachers and pre-service teacher candidates has been expressed for some time. Abramovich and Brouwer [7] refer to the 'hidden contract' [8], an institutional structure between teacher and students in which students are expected to sit quietly, do the work of emulating the teacher's examples and refrain from asking non-routine questions or displaying off-task behavior. The latter, in the context of the hidden contract, is never considered to occur when students are bored or unable to perform expected tasks. Abramovich and Brouwer [7] provide examples of curricular activities that challenge teachers or teacher candidates to re-write their hidden contracts. Their examples include mathematical problems that interconnect mathematics strands and delve into much more complex concepts than those that are traditionally taught in elementary or secondary classrooms. Our giant triangle activities are designed to challenge the traditional hidden contract that many participant teachers who attend our sessions may still embrace.

2.2 The Van Hiele Model of Geometric Thought. The learning and instructional issues in the workshop will also be related to the Van Hiele Model of Geometric Thought initially through formal introduction to the Van Hiele visual, descriptive and relational levels (see [4]: p. 53). The model has five sequential levels; visual, descriptive, relational, deductive and rigor. The last two concern systems of theorems and comparing different geometries such as Euclidean, spherical and hyperbolic. Here is a brief summary of the first three levels in the Van Hiele Model:

Visual Level: The student
- identifies, compares and sorts shapes on the basis of their appearance as a whole.
- solves problems using general properties and techniques (e.g., overlaying, measuring).
- uses informal language.
- does NOT analyze in terms of components.

Descriptive Level: The student
- recognizes and describes a shape (e.g., parallelogram) in terms of its properties.
- discovers properties experimentally by observing, measuring, drawing and modeling.
- uses formal language and symbols.
- does NOT use minimum sufficient definitions. Lists many properties.
- does NOT see a need for proof of generalizations discovered empirically (inductively).

Relational Level: The student
- can define a figure using minimum sufficient sets of properties.
- gives informal arguments, and discovers new properties by deduction.
- follows and can supply parts of a deductive argument.
- does NOT grasp the meaning of an axiomatic system, or see the interrelationships between networks of theorems.

The activities will illuminate how development through the levels may progress in the context of polyhedron attributes using guided discovery and whole group discussion. This is in accordance with the instructional cycle described by the Van Hiele model [4].

3 Instructional Strategies

The following instructional strategies will be incorporated into the workshop. Teachers who embrace traditional instructional approaches often begin to undergo belief changes when engaged in these activities. This increases the potential for them to rewrite their 'hidden contracts' about their own classroom norms.

3.1 The instructional cycle of guided discovery and instructor lead discussion. In accordance with the Van Hiele instructional cycle [4], this starts with a discussion of the students′ existing knowledge of concepts and formal vocabulary as the instructor sets the context for the activities. The instructor then provides instructional activities in which students explore and discuss concepts, preferably within small groups, and come to a consensus about the concept, or complete the task. The instructor's role is to facilitate, provide hints, and ask scaffolding questions rather than to provide answers. Students should construct their own solutions and knowledge from their own thinking rather than rely on the teacher for direct information. After the class has completed the activities, the teacher leads a whole class discussion. Each group shares its findings, and, through discussion, misconceptions are re-conceptualized. Even if groups share the same conclusions, much can be gained when students hear explanations in different words or from slightly different viewpoints.

3.2 Instructions are given in terms of feature properties. We may say 'make a shape with 5 triangles at each vertex' rather than showing an example and saying 'make one that looks like this'. The difference is that in imitating a shape students/participants use visual cues of symmetry, number and so on without paying specific attention to them. With our instructions, the students/participants may not know in advance what the shape will look like, and so will have to check local properties to complete the task. Trial and error is usually involved (figures 4, 5 and 6). When a problem emerges participants must negotiate how specific faces, edges and vertices should be changed to fix it, whilst attempting to minimize the amount of back stepping. In terms of the Van Hiele Model, this promotes development from the visual to the descriptive level in that participants actively internalize properties of the particular figure that they are creating.

3.3 Construction using flexibly connecting large scale materials. This forces people to work together to hold, build and adjust shapes, leading to negotiation between students/participants. Also, as it may be physically impossible to see the whole shape while working on one part, errors are made. Active negotiation, usually including reasoning and justification helps participants solidify their Van Hiele descriptive level knowledge about the particular figure.

3.4 Discussion of definitions and specific properties. Instructor led discussion of shapes (figures 2, 6 and 9) brings out observation and awareness of details within each shape (descriptive level). Comparison of figures leads to development of the relational level. For example, which shapes have parallel faces (figure 7), or have antipodal faces or vertices? This provides an opportunity to discuss definitions of faces, edges and vertices in polyhedra as compared to polygons. Note that different terminology may be used in different educational systems; edges of polygons may be called sides, and vertices may be called corners.

There is scope to ask participants to give as many differences as they can find between constructed shapes; for example (section 4.2) comparing pentagonal based pyramids with triangular (figures 2 and 3) and with square based pyramids. Features to be mentioned here include height, base shape, base area, slopes of sides or edges, base width as well as how many triangles meet at the top vertex. This requires students/participants not only to see that the shapes look different, but also to identify and describe specific properties that are different. Requiring students/participants to suggest and justify the differences themselves rather than have the instructor list them has a beneficial impact on our goal to rewrite their traditional 'hidden contract'. Alternate definitions of regularity using different properties (section 4.4) can also be discussed, which promotes development of geometric thinking at the relational level.

3.5 Advantages of large-scale models for discussion and observation. Everyone can get a clear view of a shape and see and touch its features. Shapes can be held by an individual edge or vertex, and a whole hand can touch just one face. This means the discussion can clearly focus on, and involve ideas or questions about, featured details and how they inter-relate. The scale also enables complex constructions to be made clear, in visual and tactile ways, as shown in figures 2 and 6 to 9.

The scale of the shapes also means that as you get closer, or inside, or further away, their appearance may change, which can emphasize details and relationships of the parts to the whole. Figure 7 shows someone sitting inside an icosahedron. The way a shape is positioned in space also emphasizes different features. Figures 6 and 7 show the same polyhedron with central antipodal vertices top and bottom (figure 6) and with central antipodal faces top and bottom (figure 7).

Figure 10:
Turn and stop game

This provides an opportunity to demonstrate rotational symmetry about the observed vertical axes going through vertices, edge midpoints, centers of faces, and the centers of the whole shapes. The turn and stop game (figure 10) is an interactive way to investigate the rotational symmetries of polyhedra. The polyhedron is rotated from an initial position, by students walking around with it, until it reaches the point where it looks the same and everyone shouts 'Stop!' Reflective symmetries are best shown by holding the shape so the plane of symmetry is either horizontal (like reflection in water) or vertical (like on a person's face). For example with a tetrahedron, the former is shown by holding it with one edge vertical and the latter when it is positioned with one pair of opposite edges horizontal on the top and bottom.

3.6 Encouragement of students/participants to ask their own questions and build their own shapes. The way in which the assembly process requires them to work out how to construct and negotiate can give a sense of ownership of the activity which permits participants to think of their own goals or questions. Questions from previous participants have included: 'Does there need to be at least a 180º angle on a vertex so it is 3-dimensional when it closes up?' (see figures 1 to 3: Angle Sum), and 'Is every antiprism always the center of a [pyramid]?' (as the octahedron is in the center between four tetrahedra, filling a double-edge length tetrahedron, figure 8).

The participants′ personal investment in solving construction problems in the less traditional classroom setting gives them a stake in deciding what to do next such as in the open-ended artistic and mathematical curiosity-driven discovery activity (figure 11). The availability of a physical manipulative model in space which is readily constructed into a wide variety of shapes facilitates exploration of a wider range of shapes. A description is given in [1] of activities in an elementary school which successfully combined the play aspect of the giant triangles with the mathematical concept explorations that the instructors overlaid.

Figure 11: *Curiosity-driven discovery; 'we wanted to see how the shapes fit together'*

4 Mathematical Content of the Activities and Extension Activities.

The following activities all extend participant's knowledge toward the van Hiele relational level in that they compare and contrast a variety of different, but related, figures attending to particular properties.

4.1 Polygonal regularity. The giant triangles are introduced with emphasis on them all being the same size and shape; equilateral triangles having equal length sides and equiangular in having equal angles. Note that a square is neither the only equiangular quadrilateral nor the only equilateral quadrilateral.

4.2 Angle sum and pyramids. Figures 1, 2 and 3 give snapshots of activities that develop a transition from polygons to polyhedra when 6 triangles are placed around a point on the floor and then connected together. When this is repeated with less and less triangles, a set of pyramids is created as 5, 4 and 3 equilateral triangles are joined together around a point and placed upright on the floor as in figure 3. As mentioned above (section 3.4), participants are then asked to identify distinguishing properties of regular-based right-pyramids (symmetrical), including height, base shape, base width, base area, volume, slope of faces, slope of edges etc. This also focuses on the 'angle sum' at an individual vertex. Angle sum at a vertex is the angle swept by the faces of the polyhedron touching the vertex, and is clearly seen when the vertex is opened up and laid flat in the floor. For a regular tetrahedron it is 180º (three equilateral triangles) at each vertex, for a cube, 270º (three squares) at each vertex, and so on.

If the angle sum is less than 360º, then the vertex will become three dimensional, in that it will not lay flat on the floor when the faces are connected, unless faces are folded up together flat. When only two faces are used they must join together back to back and a degenerate case is created. Vertices with an angle sum greater than 360º are shown in figure 9 on the stella octangula. Situated in the centers of the faces of the circumscribing cube, they have 8 equilateral triangles joined at a vertex, giving an angle sum of 480º. Further extensions that explore angle sum more deeply for higher grade levels are given in [2] (lessons 5 and 6).

4.3 Volume and space filling. A possible extension activity is to quantify the volumes of the pyramids. This can be done at lower levels by direct measurement and at higher levels through the use of formulae, the Pythagorean Theorem and trigonometric ratios. An intermediate level approach using volume scaling (see [2], lesson 1) is given by the tetrahedral octahedral space filling decomposition of a doubled-edge tetrahedron. The central octahedron (figure 8) is comprised of two square pyramids. Figures 8 and 9 show configurations facilitated by the lecturer to demonstrate ways in which polyhedra can fit together. Figure 11 shows students trying their own experiment to see how polyhedra fit together.

An additional activity that focuses on space filling uses parallelepipeds. A parallelepiped can be constructed by joining a tetrahedron to each of two opposite faces of an octahedron. It will then have six faces that are rhombi, each composed of two coplanar triangles. If many are produced, they can be stacked to demonstrate space filling. As each is made up of an octahedron and two tetrahedra, this demonstrates that octahedra and tetrahedra together fill space in a 1:2 ratio. They can also be viewed as cubes which have been deformed, giving a connection between space filling by cubes and space filling by octahedra and tetrahedra.

4.4 Definitions of regularity using number and symmetry. Figures 4 and 5 show trial and error in making ‘a polyhedron with 5 triangular faces at each vertex’. This type of instruction for each Platonic solid emphasizes certain regularity properties of the Platonic solids and differences between them: the same number of identical regular polygonal faces at each vertex for each solid. Discussion of properties can be developed into discussion of definitions.

As an extension, alternative definitions of regular polygons and polyhedra, in terms of minimum sufficient conditions, can be explored. For example ‘4 identical equilateral triangular faces at each vertex’ is sufficient to define a regular octahedron. Alternatively, in terms of rotational and reflective symmetries of the whole shape, we can, for example investigate if we can define a regular tetrahedron as ‘a polyhedron having 3 and only 3 planes of reflectional symmetry through each vertex and the center of rotation of each face’.

In general the reflective and rotational symmetries of a Platonic solid are as follows. Consider a polyhedron where at each vertex, m regular polygonal faces come together, each having n edges or sides. There are n, 2 and m planes of reflection through each face midpoint, edge midpoint and vertex respectively. Also there are rotational symmetries of order n, 2 and m through each face midpoint, edge midpoint and vertex respectively. A rotation of order n is a rotation through an angle of $(360/n)^{\circ}$. All planes of reflective symmetry and axes of rotational symmetry pass through the center of the shape.

Definitions of polyhedral regularity in terms of face shape and total numbers of faces and edges can be seen to work for some Platonic solids, such as the tetrahedron (‘4 equilateral triangular faces’ or ‘6 edges with equilateral triangular faces’). This also works for the cube and dodecahedron, but not for the octahedron or icosahedron. Irregular polyhedra can be made with any even number of equilateral triangular faces greater than 4. We see below (section 4.5) why we cannot make a polyhedron with just an odd number of triangular faces.

4.5 Counting. Figure 6 shows the exploration of strategies to count faces and edges using subdivision and rotational symmetry of the icosahedron. The top vertex is the center of a cap of 5 triangles. Opposite it, on the floor is the bottom vertex in the center of another cap of 5 triangles. The remainder of the icosahedron is a central ring containing 5 triangles pointing up and 5 pointing down. This gives a total of 20 triangles and the way they form groups of 5 (top cap, bottom cap, central ring faces pointing up and central ring faces pointing down), highlights the rotational symmetry about a given vertex.

Consider what happens to the faces when this icosahedron is rotated by 72º (1/5 of a whole turn, i.e. (360/5)º) about the vertical axis through the top vertex and bottom vertex (figure 6). Each face will be rotated to a different face within its forementioned group of 5. If this is repeated then each face will cycle through all the other faces in the group and then, on the fifth, rotation return to its original position. The total number of faces, made up of groups of 5, must therefore be divisible by 5. This illustrates a general counting rule:

If a rotation of a polyhedron, about a specific axis by a specific angle in a given direction, moves each face to a different face and also all faces first get back to their original positions after n of these rotations, then the faces form groups of n and so the total number of faces will be divisible by n.

However this does not apply with a 1/3 rotation (120º) of the icosahedron about an axis through the center of rotation of a face. Even though it is a rotational symmetry of the whole shape, it does not move each face to a different face. We can apply the rule using vertices or edges instead of faces, for this rotation. The icosahedron has 20 faces (divisible by 5), 30 edges (divisible by 5 and 3) and 12 vertices (divisible by 3). The counting rule can be verified for each of these numbers using different rotational symmetries.

A subtlety emerges when counting edges: each edge of a polyhedron is shared by two adjacent faces. If we start to count the edges of an icosahedron by saying ‘there are 3 sides for each of the 20 triangular faces, and we get 3x20=60’. However, the sharing means we also have to divide by 2 to obtain 30 edges. An extension activity is to count the numbers of faces and edges of as many polyhedra with triangular faces as possible to lead up to and verify the calculation. We can observe that multiplying a whole number by 3/2 in this way does not give a whole number of edges if we start with an odd number of faces. This leads to a proof that all polyhedra with just triangular faces must have an even number of faces, because a polyhedron cannot have a half edge (see [2], lesson 4).

4.6 Archimedean solids, stellation and duality. The polyhedra known as the ‘Archimedean solids’ contain faces of more than one type of regular polygon, such as the squares and triangles that make up the faces of the cube octahedron. The edge skeleton of a cube octahedron can be constructed with the giant triangles, using 6 square base pyramids to create the squares. If the pyramids point inward, then the cube octahedron's structure is visible. See also the endo-pentakis-icosi-dodecahedron constructed this way [3].

The class of stellated Platonic and stellated Archimedean solids can all be constructed with the giant triangles. In this case, all the pyramids point outwards, and their bases form the faces of the underlying polyhedron. For example, 12 pentagonal based pyramids can be connected to make the stellated dodecahedron. In the case of the stella octangula (figure 9), an octahedron is stellated and the outer vertices are connected with tape to form the edges of a cube. Each of the 8 pyramids added has an apex corresponding to a vertex of the cube, and a base corresponding to a face of the octahedron. This illustrates the duality of the cube and octahedron. The one-to-one correspondences between the faces of one polyhedron and the vertices of the other, and the one-to-one correspondence between the edges of each polyhedron, constitute the duality relationship between the two polyhedra.

5 Conclusion

We present activities and instructional strategies that exploit the capabilities of the giant triangles to bring out a range of geometry concepts, and promote the development of geometric thinking. In terms of the

Van Hiele Model of Geometric Thought, the development of thinking through the visual, descriptive and relational levels are promoted. The activities and instructional strategies also encourage candidate or practicing teachers to rewrite their student-teacher ‘hidden contracts’ to promote more effective learning in the classroom.

6 The Authors

Simon Morgan has a PhD in Mathematics from Rice University and many years experience providing courses with in- and pre-service teachers at all levels with Rice University and the University of Minnesota. Jacqueline Sack has a doctorate in Mathematics Education from the University of Houston and is now a faculty member at University of Houston-Downtown, Department of Urban Education. She has many years experience as a middle school mathematics teacher, school district mathematics coordinator and provider of in-service teacher development. Eva Knoll has a doctorate in Education from the University of Exeter and is now a faculty member in the Faculty of Education of Mount St Vincent University, Halifax, Canada. She conceived and designed the giant triangles.

7 References

[1] Eva Knoll and Simon Morgan. *Preliminary Field Explorations in K-6 Math-Ed: the Giant Triangles as Classroom Manipulatives.* In Sarhangi, R., (Ed.), Bridges: Mathematical Connections between Art, Music and Science, pp. 205-212, 2002.

[2] Eva Knoll and Simon Morgan. *Polyhedra, Learning by Building: Design and Use of a Math-Ed. Tool.* In Sarhangi, R., (Ed.), Bridges: Mathematical Connections between Art, Music and Science, pp. 331-338, 2000.

[3] Eva Knoll and Simon Morgan. *Barn-Raising an Endo-Pentakis-Icosi-Dodecahedron.* In Sarhangi, R., (Ed.), Bridges: Mathematical Connections in Art, Music and Science, pp. 131-136, 1999.

[4] Pierre M. Van Hiele. *Structure and Insight: A Theory of Mathematics Education*. Academic Press 1986.

[5] Lee Shulman. Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15* (2), pp. 4-14, 1986.

[6] Deborah L. Ball & Hyman Bass. *Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics.* In Jo Boaler (Ed.), Multiple Perspectives on Teaching and Learning (pp. 83-104). Westport, CT: Ablex Publishing, 2000.

[7] Sergei Abramovich and Peter Brouwer. *Hidden Mathematics Curriculum: A Positive Learning Framework.* For the Learning of Mathematics, 26 (1), pp. 12-17, March, 2006.

[8] Timothy F. Slater. *When Is a Good Day Teaching a Bad Thing?* The Physics Teacher. Vol. 41, pp. 437-438. October, 2003.