A Workshop in Geometric Constructions of Mosaic Designs

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Abstract

The focus of this workshop is to study two different approaches for creating mosaic designs and to compare them. A traditional and well-known method in this regard is using a compass and straightedge. The other method that will be introduced and discussed during this workshop is the use of the modularity method. Modularity is a special cutting and pasting process of tiles to create tile designs. During this workshop the participants will create a series of designs using a compass and straightedge, and then through some hands-on activities, they will discover that the same designs could be constructed using modularity. The cuts in the modularity method are elementary in that they cut the tiles (mainly squares) along line segments, such as the diagonals, to divide the tile into two or three pieces. For certain designs, there are more detailed methods of cutting that the participants should execute to construct more complex geometric shapes.

1. Introduction

The polygonal sub-grid system, which is based on extensive use of compass and straightedge geometric constructions, is the main method in medieval Persian art, which has been well-documented [1]. Artists and craftsmen used this method widely— which exhibited their highly developed skills in geometry or presented their collaborations with geometers of their times. Nevertheless, it would be a mistake to assume that one, and only one, method was responsible for all the ornamental patterns and tiling designs of Persian art. The modularity approach, based on color contrast of cut-tiles, may be considered as another possible method used by these artisans.

Figure 1: *Mathematics education major students in a class activity in modularity.*

During the workshop, after analyzing the photographs of some existing mosaic patterns, the participants will use a compass and straightedge to create those designs and transfer the patterns to grid papers. They practically construct the layout for the tessellation of each of the tile designs. After that, the participants will construct these patterns using modularity techniques applied to simple single-color square-shaped tiles. They will realize that the modularity method is more elementary and simpler than the compass and straightedge construction. This will show the possibility that the original layouts of these designs, which are much older than the existing structures in the photographs, were discovered using modularity, centuries before their reconstructions using a compass and straightedge.

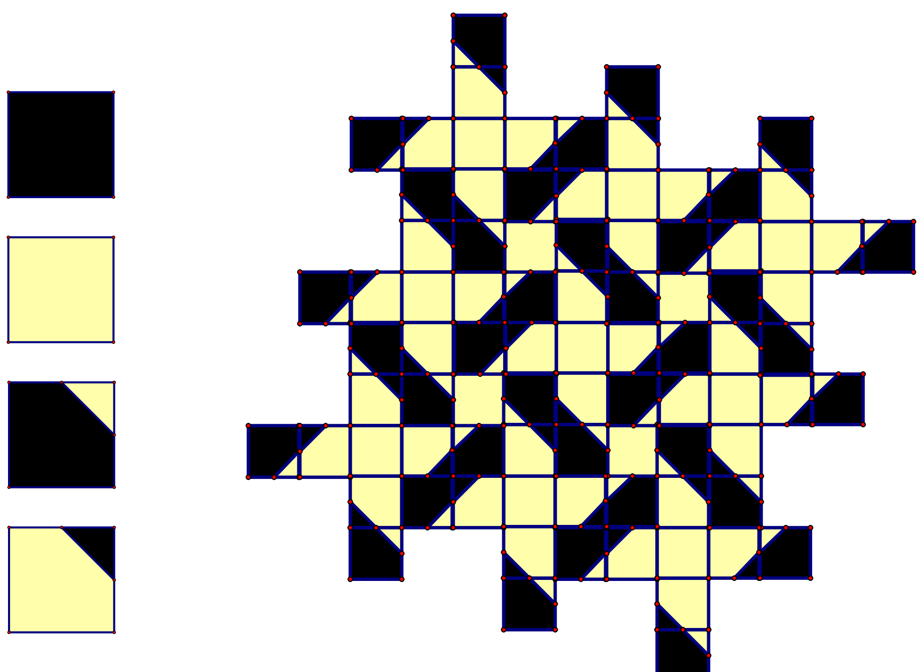
Section 2 presents modularity via one example. Section 3 shows the constructions of some patterns through both compass and straightedge, and modularity methods. Section 4 demonstrates cuttings of square tiles for the construction of *octagram and cross* design patterns.

All the illustrations in this article are by the author using the Geometer's Sketchpad software utility.

2. Modularity in Brief

By a "modularity" approach, we mean a method that uses the cutting and pasting of two different colored tiles to create a set of two-color modules. Here, cutting means breaking a tile into two pieces along a single line segment with the endpoints on the edges of the tile.

For example, cut both a solid black tile and a solid yellow tile along a segment that connects the midpoints of two adjacent sides, and then exchange the pieces. This will result in two two-color "modules" where one is the negative of the other. Now, including the two original single-color square tiles, there are four modules to work with to create new tessellations (Figure 1.a). A tessellation created using this set of modules is shown in Figure 1.b. For more information about modularity please see [2-3].



(a) (b)

Figure 2: (a) *Four modules created based on two different colors of congruent squares,* (b) *A mosaic pattern created from this set of modules.*

3. Hat Pattern, Maple Leaf Pattern, and More

The pattern in the 14th century Iranian bowl in Figure 3.a [4] has been named the *hat* pattern in some literatures. An earlier document for this pattern is on the western tomb of a pair of 11th century tomb towers in Kharraqan, in western Iran (Figure 3.b). The most interesting phenomena about these towers (unfortunately one of them has partially collapsed in recent years) is the fact that the exterior surfaces are entirely covered with geometric patterns, executed simply by the selective cutting and placement of bricks set with mortar [5].

(a) (b)

Figure 3: (a) *A 14th century Iranian bowl*, (b) *the 11th century western tomb tower in Kharraqan, Iran. Photography courtesy of Ann Gunter.*

Figure 4 illustrates steps taken by a geometer or a highly skilled artisan to compose the *hat* grid using a compass and straightedge. Interested readers may find similar constructions in [6] and [7].

During the workshop, the participants will take these steps to construct the *hat* tile design. Then using the tiles they will construct tessellations.

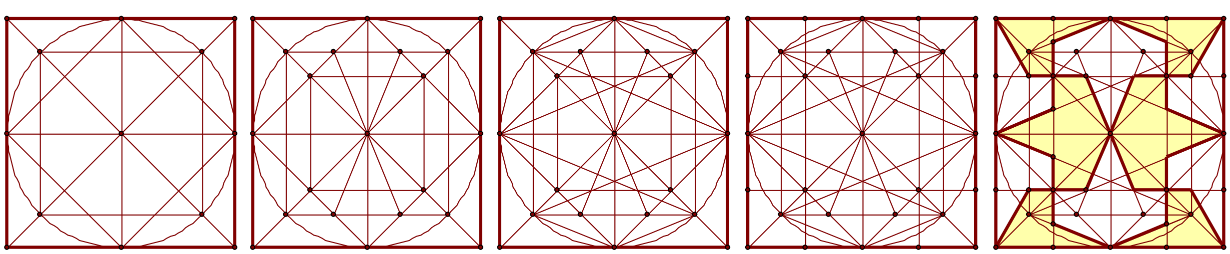


Figure 4: *Polygon construction approach for generating the grid for the "hat" tiling.*

The following figure presents the way that the *hat* pattern can be generated by only two opposite modules (without the use of original single color tiles). The cut is from the midpoint of a side to the vertex of the opposite side.

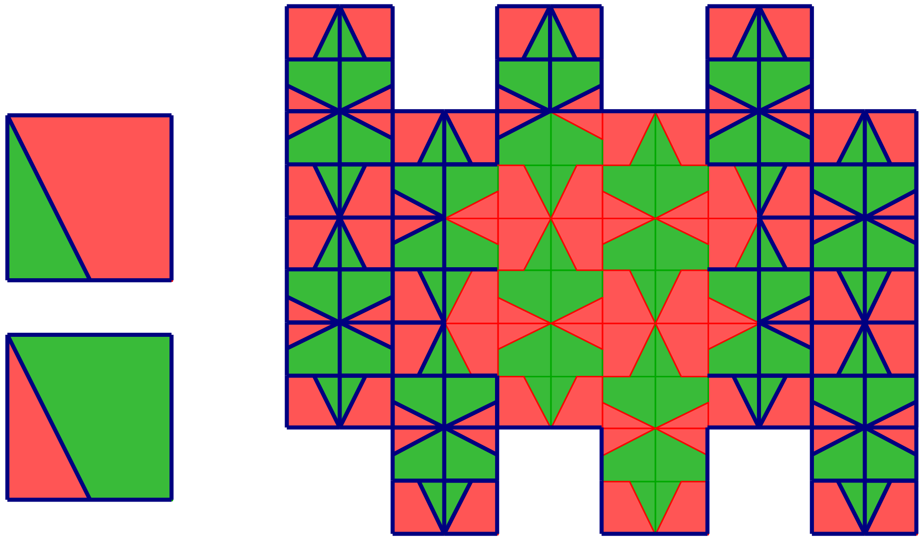


Figure 5: *The hat modules and their tessellation.*

Another pattern appropriate for this workshop is the *maple leaf*. A motif for this pattern can be shaped using an isosceles right triangle to construct its tessellation. The isometries involved in the process of creating the motif are "a quarter rotation of one side to another" and the "reflection of the shape under the hypotenuse" (Figure 6). The mathematical notation for this pattern is *P4m.* It belongs to the square lattice of wallpaper patterns and its highest order of rotation is 4. It can be generated by 1/8 of its square unit. If we consider this design as a two-color pattern, then its crystallographic group classification will be *p*4'*g'm* (2-fold rotational symmetry, vertical and horizontal reflective symmetries).

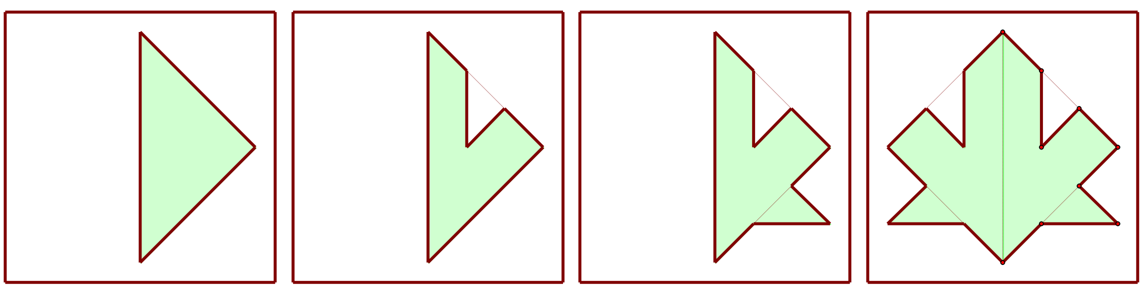


Figure 6: (From left to right) *An isosceles right triangle, modification of one side, rotation to the other side, reflection along the hypotenuse*.

The following figure shows the process for a traditional compass and straightedge construction.

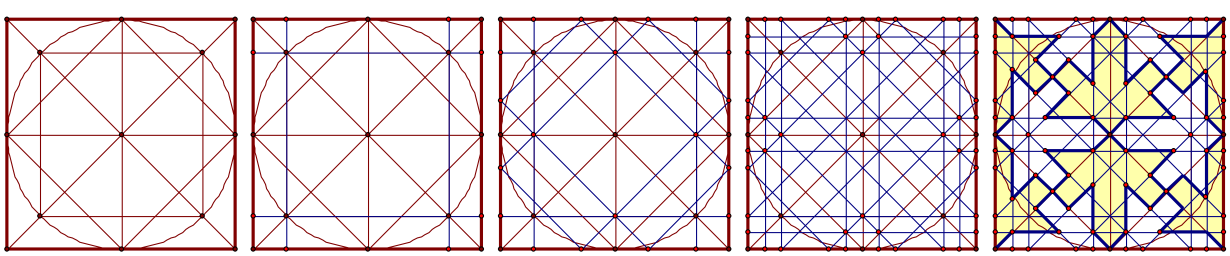


Figure 7: *Traditional construction of the "maple leaf" pattern.*

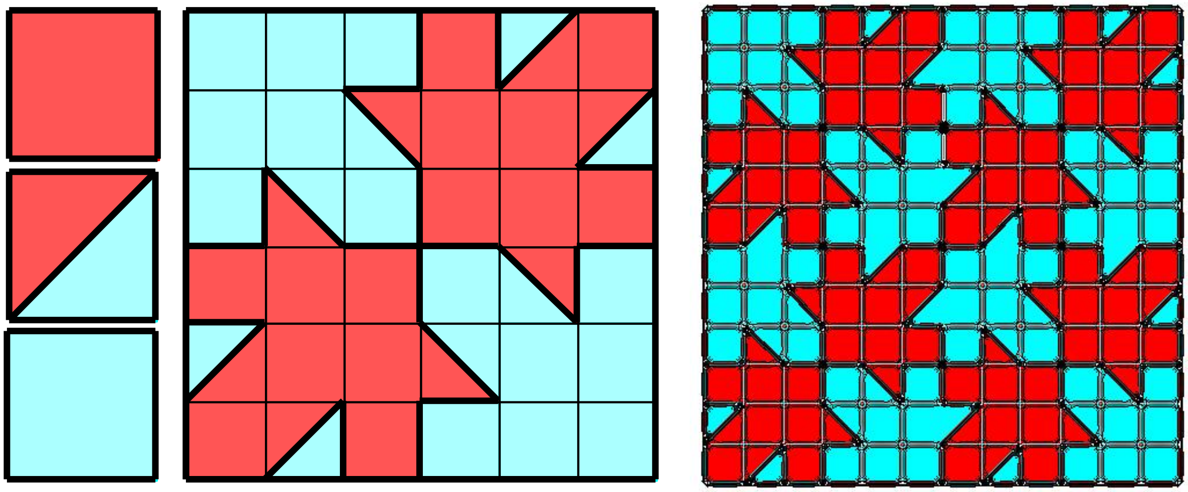


Figure 8: *Generating the "maple leaf" tessellation using three modules.*

The workshop participants will use a set of three modules, created from a diagonal cut on two different colors tiles, to make the *maple leaf* tessellation (Figure 8).

Figure 9 shows an existing pattern on the wall of *Mossalâ*, Herāt, Afghanistan. The traditional compass and straightedge construction of the motif for this pattern is shown in Figure 10. The construction of this pattern using modularity is illustrated in Figure 11.

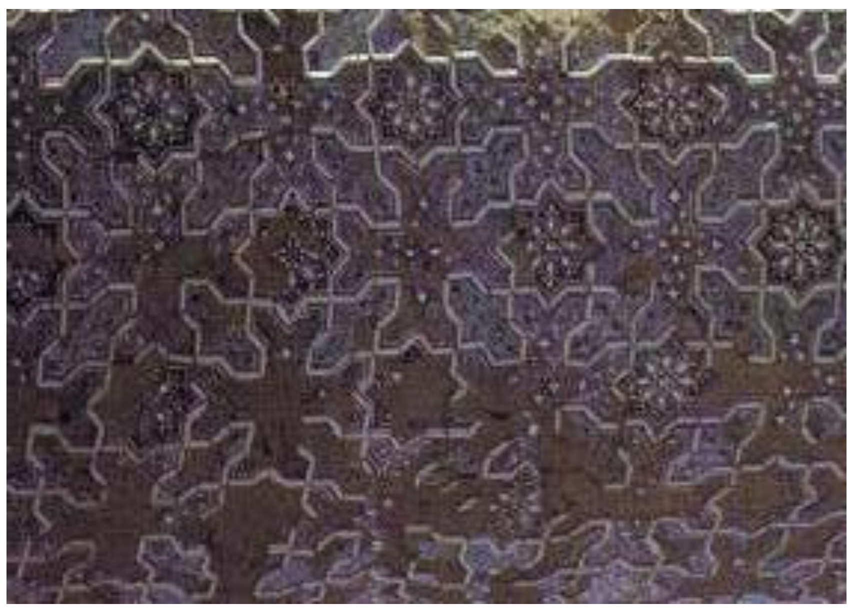


Figure 9: *An existing pattern on the wall of Mossalâ, Herāt, Afghanistan.*

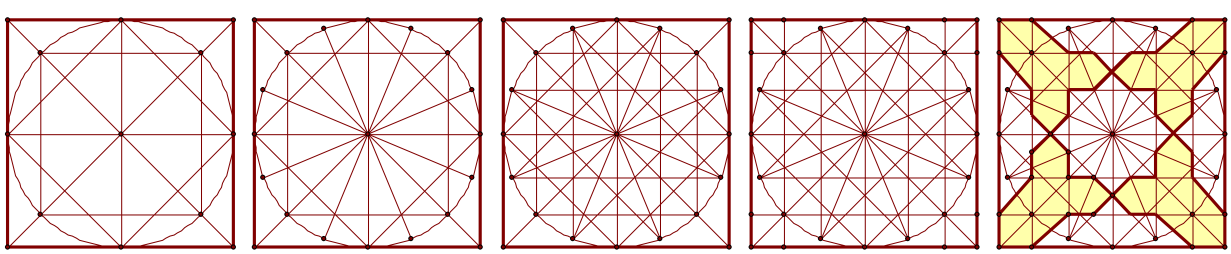


Figure 10: *The compass and straightedge construction of the pattern in Figure 9.*

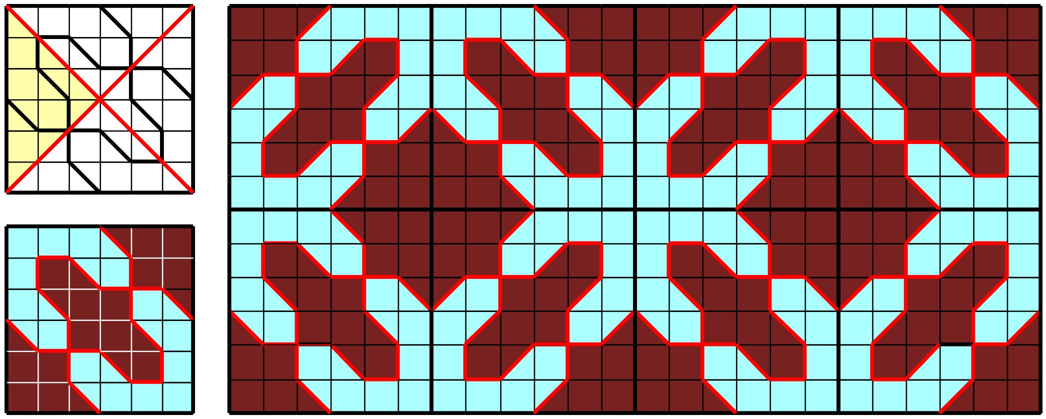


Figure 11: *The mosaic design construction in Figure 9 using the modularity technique.*

4. More Modules based on Complex Cuttings of Squares

The photograph of the tiling in Figure 12 is from a wall of *Arge Karim Khani* Fortress in Shiraz, Iran. Figure 13 shows the way that the underling pattern can be constructed using a compass and straightedge.

Figure 12: A mosaic pattern from a wall of *Arge Karim Khani* Fortress in Shiraz, Iran.

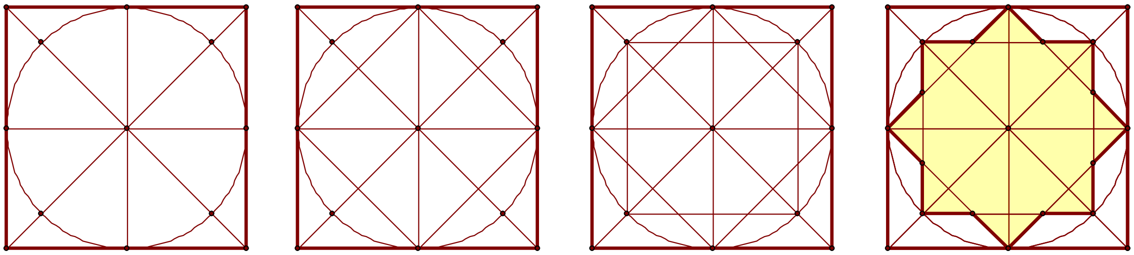


Figure 13: *The compass and straightedge construction of the pattern in Figure 12.*

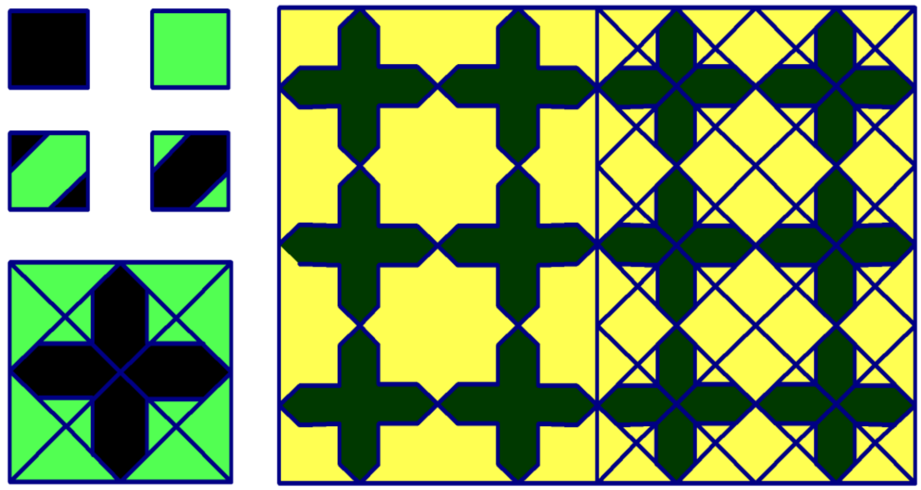


Figure 14: *The modularity approach to approximate the tiling in Figure 12.*

The set of modules in Figure 14 have been created based on a set of cuts from the midpoint of a side of the tile to the midpoint of the adjacent side. Using this set we are able to generate an octagram-cross tiling. However, this set does not generate an octagram with congruent sides. So Figure 14 does not demonstrate the exact outline of the tiling in Figure 12. In fact, the octagram created using these modules has sides with two different lengths of 1/ 2 and $\sqrt{2}/2$. Another set of cuttings of square tiles are needed to replicate "octagram and cross" design patterns to their exact dimensions.

The following is a solution to this modularity problem: Let *ABCD* be a square tile with sides congruent to one unit (Figure 15). In order to create an equilateral octagram using modularity, this square should be cut in a way that the non-regular hexagon *AFGCHI* becomes equilateral. Suppose that the sides of the hexagon are congruent to *a* units. Let $\overline{FB} = b$ units. Then $a + b = 1$ unit (I). Moreover, since ΔBGF is an isosceles right triangle, we obtain $a^2 = 2b^2$ (II). The above equations (I) and (II) result in $b = \sqrt{2} - 1$. So to make the correct cut, construct an arc with center *A* and radius $\overline{AC}$ to cut ray $\overrightarrow{AB}$ at E ($\overline{AC} = \overline{AE} = \sqrt{2}$). Construct a second arc with center *B* and radius $\overline{BE}$ to cut the sides of the square at *F* and *G*. To construct the rest of the module in the tile is now straightforward.

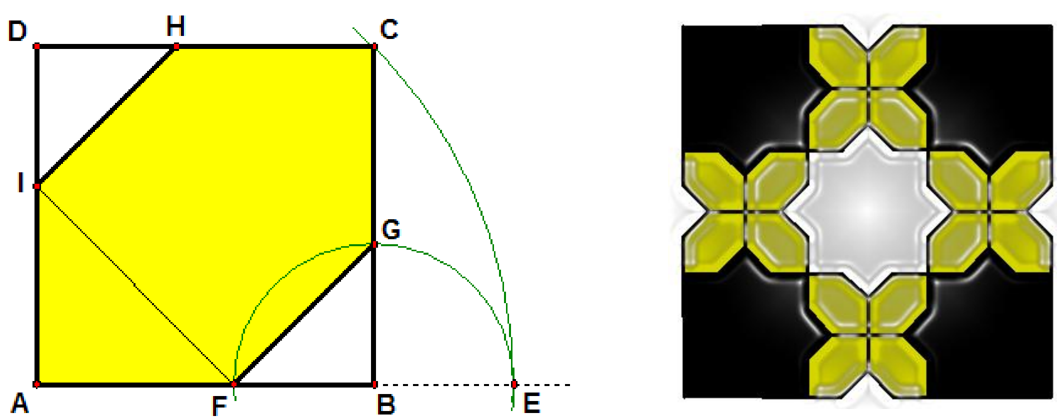


Figure 15: *The cutting instruction for a perfect "pentagram and cross" tiling using modularity, and its tessellation.*

5. Conclusion

Topics in geometric constructions bring excitement to mathematics classrooms and will improve students' attitudes toward this subject. The modularity workshop – along with a number of other workshops in mathematical connections in art and culture – has been a part of activities in a seminar course at Towson University offered during the 2009 fall semester by the author. This course serves as a capstone to the degree program in mathematics education. The course is designed to integrate the mathematical knowledge and pedagogical issues in secondary schools. A student wrote "As was evident in our class, simple use of a set of toys will motivate and engage students. Because the students were actively participating with both mind and body, the content of the lesson was acquired more easily and will probably be retained for far longer."

Figure 16: *The end result of the modularity activity in Figure 1, which was based on a tilling on the wall of a Persian mausoleum.*

References

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