Projection of Point Sets to a Lower Dimension with Applications in the Arts

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Abstract

The paper deals with projections that map points between sets represented in different dimensions, in particular we consider examples of such projections that could have applications in the arts. Often Art involves projecting higher, in most cases three-, dimensional objects into two-dimensions. An example with a wide application is when one makes a correspondence between the points of the 3D "color cube" and the points of a 2D (plane) picture. We use this idea to present a decomposition of color pictures into symmetric and antisymmetric components.

An obvious example occurs when one maps 3D objects into a 2D surface. I note that artistic images do not consist of a continuum of points in the 2D plane, but rather a countable, finite manifold of pixels. On the other hand, the spectator's eyes are unable to perceive a continuum manifold of pixels. Our visual apparatus records distinct pixels in three different basic colors, individually, and we compose an image from these three sets of finite units of information [3, p. 27]. These pixels are not points in a mathematical sense (i.e. without extension). A region in 3D space (a closed volume) actually contains more points (i.e. small cubes) in this sense of having extension than a region in 2D space (i.e. a shape circumscribed by a closed line in the plane). Although mathematically both sets contain the continuum infinity of points, in everyday situations concrete material objects consist of a set of countable, discrete points (or small topological environments of those points). In a physical sense, matter consists of discrete units: atoms, molecules, crystal cells, etc. and in an artistic sense the depictions (of material objects or simply of the artist's fantasy) consists of pixels (or, at least, our eyes can only perceive them in discrete pixels).

Before presenting the mapping of points of a "color cube" into the plane, I will illustrate my presentation with a few works of a Russian contemporary constructivist artist, Aleksander Pankin. Pankin is an admirer of the art of K. Malevich. He has reconstructed a few plane works of Malevich in the form of an affine projection of three-dimensional objects into the plane. I will present one of these objects (from the permanent collection of the Mobil MADI Museum), and two other objects in photographs provided by the artist. These artworks illustrate two possible consequences of mapping from higher dimensional spaces. (Pankin projects from 3D to 2D. Most works of T. F. Farkas are examples for projections from higher dimensions to 2D.) Although in practice Pankin uses a torch light, in principle he projects with a parallel beam. (This means he does not apply perspectivic projection. For perspective, as a combined symmetry transformation see [4].)

Figure 1: *A. Pankin:* Three spatially arranged black squares *whose projection from the direction marked by the red line on the floor produces the shape of Malevich's* Hammer.

(a) Projection of a three-dimensional object into the plane can produce different planar objects depending on the direction of the projecting light-beam. For example, when Pankin produces Malevich's *Hammer* on the plane of a wall in the form of his 3D object's shadow, it works only from a single direction of the projecting light-beam. If one moves the direction of the light-beam, the projected image will change. The two-dimensional artwork of Malevich is a special case of possible projections of the object realized in the real three-dimensional space.

(b) A given projected two-dimensional image can be produced by the projection of different three-dimensional objects. For example, when we analyze Malevich's *Hammer* as a projected 2D image, it can be produced either (b1) with the light-beam from the same direction through other 3D objects, whose cross-section perpendicular to the given direction of the light-beam coincides with that of Pankin's artwork, or (b2) with light beams from different directions projecting certain other shaped 3D objects.

Figure 2: *A. Pankin*: Projections of triangles.

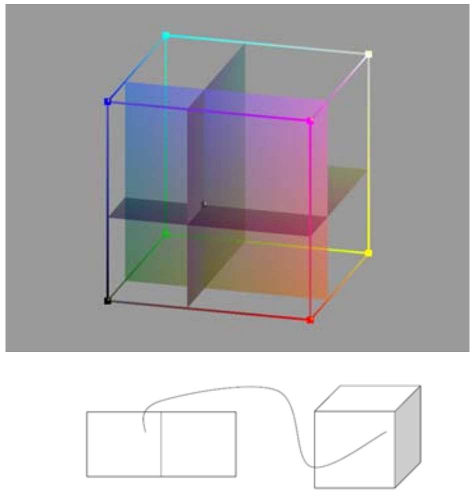
(b1) In the first case – when the direction of the light-beam is fixed – the artist has a great flexibility to shape his object taking care only of the cross-section to be left intact.

(b2) In the latter case, each of the shapes can be smoothly deformed, and simultaneously rotated and translated in the 3D space, so that a mathematical dependency (a function) is established between this smooth deformation plus movement, and the smooth change in the angle of the projecting light-beam. One cannot deform-and-move the 3D object and the angle of the projection independently, but if one obeys the dependence-rule that couples the changes (i) in the deformation of the shape and placement in the space of the 3D object, and (ii) in the direction of light, one may produce an invariant projected image. The number of constraints on the dependence on the two changes is lower then the number of degrees of freedom of the system, so infinitely many – smoothly changing – configurations can produce the same invariant 2D image.

The above demonstration of mapping from a 3D real space to the 2D plane provides a way to understand easier the following mapping from an abstract space to the 2D real plane for readers less sophisticated in mathematics. A similar, but abstract projection is executed in coloring pictures. Here we make a mapping between points of an abstract 3D space representation of colors and the pixels of a 2D (plane) picture. In a mathematical sense, a picture is a two dimensional arrangement of pixels. Each pixel can be characterized by a color. A color can be defined by three parameters, in other words as a weighted combination of three basic colors (either *rgb* or *cmy*). I apologize for the detailed description of the procedure below for it is known by many mathematicians and a few engineers. However it has become apparent to me in recent years that it is not obvious at all for many artists and less mathematically sophisticated scientists.

An individual color can be represented as an inner point (more precisely a minor cube) in a 3D (*RGB/CMY*) cube, whose three edges intersecting in one vertex (origin) form three perpendicular axes, the axes representing the three basic colors. The scale of colors forms a smooth continuum. This scale is artificially made discrete for computing purposes. Our computer programs divide each edge of the color cube into 256 sections. (This is a practical convention, however, we could agree in other divisions as well, e.g. 256 sections in [0, 1] scale, or unitary integer division in [0, 512], etc.) The first section is marked by the sign "0", and the two hundred and fifty sixth section is marked by the sign "255". (In this case the figures in quotation marks are names, i.e. "signs" of the individual sections, and not mathematical "values".) The distance of the "higher" end of the section denoted by the "255" sign is 256 section lengths from the origin. The origin of this 3D orthogonal system is marked by the sign triplet (0, 0, 0) and corresponds to the lack of any color, i.e. to black. The axes are marked by red (r), green (g) and blue (b), i.e. by the basic additive colors, respectively. The last section in the color cube's red edge – one of its neighboring vertices to black – is marked by (255, 0, 0), this is the red vertex. The vertex on the green axis is at the point (0, 255, 0) and on the blue axis at (0, 0, 255). The vertex opposite to the black is marked by (255, 255, 255), this corresponds to white. The remaining three vertices of the color hexahedron (cube) are the yellow (255, 255, 0), the cyan (0, 255, 255) and the magenta (255, 0, 255). Note, that these "figures" are not numbers, rather signs. However, one can make them correspond to a number. So we can handle the individual colors as co-ordinates in a "color axis" (r, g, b) reference frame. We consider this color reference frame as (a finite volume cube in) a 3D *abstract space*.

Each pixel in a (planar) picture is a result of a mapping from the cube in the abstract color space, when one of the points of the color cube – defined by the vertices in the above paragraph – is mapped onto one of the pixels (more precisely, its color) of the given picture. Note, each pixel in the 2D picture corresponds to one point in the 3D color cube, but (a) not all points of the color cube are necessarily in correspondence with a given picture, and (b) one point of the color cube may be mapped onto more pixels of the given picture.



Picture **C**olor cube

Figure 3: *Determining a point in a color cube, and mapping of the 3D matrix of colors into the 2D matrix of picture pixels.*

For simplicity, let us restrict ourselves to rectangular pictures. The pixels of a rectangular image can be arranged in a rectangular matrix. Now an addition preserving projection is established from the 3D matrix $C_{r,g,b}$ formed by the points of the three-dimensional color cube to the $P_{i,j}$ 2D matrix elements corresponding to the position of the pixels in the picture. So an element (r, g, b) of the color cube matrix $(0 \leq r \leq 255; 0 \leq g \leq 255; 0 \leq b \leq 255)$ is associated with a $p_{i,j}$ element of the matrix of pixels, where $p_{i,j}$ denotes the pixel in the i-th row and j-th column. $C_{r,g,b} \rightarrow P_{i,j}$.

The preservation of addition during the projection is not obvious, it needs demonstration. Each point in the 3D color matrix can be decomposed into the sum of two other matrix elements. $C_{r,g,b} = C'_{r,g,b} + C''_{r,g,b}$. Since each matrix element (marked by three numbers, r, g, and b) denotes a color, each (r', g', b') and (r'', g'', b'') element of the component matrices denote two other colors. We can make a correspondence between the respective elements of the C' and C'' matrices, and pixel matrices P' and P'', where P' and P'' are defined so that their sizes coincide with the size of the matrix P, and there exist correspondences between the elements $C'_{r,g,b} \rightarrow P'_{i,j}$ and $C''_{r,g,b} \rightarrow P''_{i,j}$ respectively. Thus, as the sum of the matrix elements $C'_{r,g,b} + C''_{r,g,b} = C_{r,g,b}$, the sums of the colors of the respective elements of P are made as the coverage of pixels $P'_{i,j}$ and $P''_{i,j}$. I will demonstrate in prints on transparent folios that indeed, the two component images P' and P'' produce the original pixel matrix P (Figures 5 and 6).

This mapping involves a series of interesting properties of a picture. In the following, I will present and illustrate a procedure (® 2003, [1]) and its consequences that have not been published in English, and even the Hungarian and German versions did not disclose certain details.

This procedure makes it possible to decompose any picture that has at least one symmetry axis (straight or curved) into a symmetric and an antisymmetric picture, so that the two composite pictures when added produce the original picture. I will demonstrate in prints on transparent folios that indeed, the two component images P^S and P^A produce the original pixel matrix P. An antisymmetric picture is defined so that the mirror-arranged pixels display complementary colors, where the complementary color of (r,g,b) is defined as (255-r, 255-g, 255-b). The decomposition is based on the mathematical observation that any rectangular matrix can be decomposed into the sum of a symmetric and an antisymmetric matrix – with a couple of tricks used for the concrete realization of this projection applied to colors mapped into the plane.

Let's make the mapping in the opposite order: $P_{i,j} \rightarrow C_{r,g,b}$. This means that the red component of the pixel $P_{i,j}$ is $C^{i,j}{}_{r}$, the green component of the pixel $P_{i,j}$ is $C^{i,j}{}_{g}$, and the blue component of the pixel $P_{i,j}$ is $C^{i,j}{}_{b}$. We take a concrete picture and determine the corresponding matrix elements $C^{i,j}{}_{r}$, $C^{i,j}{}_{g}$, $C^{i,j}{}_{b}$, in C assigned to each $P_{i,j}$. In this case each pixel of the picture can be characterized as $P_{i,j} \rightarrow P_{i,j}(C^{i,j}{}_{r}, C^{i,j}{}_{g}, C^{i,j}{}_{b})$.

Now let us take one of the symmetry axes of the picture P. If this axis is the vertical mirror axis, and if there are m columns in the pixel matrix, then the mirror image pixel of $P_{i,j}$ will be $P_{i,m-j}$. Its corresponding components in the color matrix will be $C^{i,m-j}{}_{r}$, $C^{i,m-j}{}_{g}$, $C^{i,m-j}{}_{b}$.

Now choose any pair of symmetrically placed pixels in P: $P_{i,j}$ and $P_{i,m-j}$. (Note, we could choose other symmetry axes, too.) The corresponding colors attributed to these pairs of pixels are $C^{i,j}{}_{r}$ and $C^{i,m-j}{}_{r}$, $C^{i,j}{}_{g}$ and $C^{i,m-j}{}_{g}$, $C^{i,j}{}_{b}$ and $C^{i,m-j}{}_{b}$. We can decompose the matrix elements in each color into elements of symmetric and antisymmetric matrices separately.

$$C_r^{i,j} = \frac{C_r^{i,j} + C_r^{i,m-j}}{2} + \frac{C_r^{i,j} - C_r^{i,m-j}}{2} = C_r^{\mathrm{S}i,j} + C_r^{\mathrm{A}i,j}$$

and similarly for the g, and b components. This means, that the colors of the symmetric $P^{S}{}_{i,j}$ matrix element, [i, j] (and according to the symmetry, also of the [i, m-j]), are $C^{\mathrm{S}i,j}{}_{r}$, $C^{\mathrm{S}i,j}{}_{g}$, $C^{\mathrm{S}i,j}{}_{b}$, while the colors of the antisymmetric $P^{A}{}_{i,j}$ matrix element, [i, j], are $C^{\mathrm{A}i,j}{}_{r}$, $C^{\mathrm{A}i,j}{}_{g}$, $C^{\mathrm{A}i,j}{}_{b}$, and the respective colors of the [i, m-j] element are (255-$C^{\mathrm{A}i,j}{}_{r}$), (255-$C^{\mathrm{A}i,j}{}_{g}$), (255-$C^{\mathrm{A}i,j}{}_{b}$).

This procedure would work, if C_r, C_g, and C_b were numbers and not signs, and if the color scales ran over all natural numbers (-∞, ∞). This is not the case. Although we can handle these signs as numbers (e.g. add and subtract them), they are interpreted on a limited range [0, 255]. The ranges of interpretation of C_r, C_g, and C_b are each [0, 255]. This coincides with the range of values of $C^{S}{}_{r}$, $C^{S}{}_{g}$, and $C^{S}{}_{b}$, but the range of values of $C^{A}{}_{r}$, $C^{A}{}_{g}$, and $C^{A}{}_{b}$ are each [-128, 127]. This not only fails to coincide with the range of interpretation [0, 255], but does not always denote "signs of color" that were interpreted only as positive numbers in the range [0, 255].

Figure 4: *Scale transformation and inverse scale transformation*.

To avoid signs that do not denote a color in the defined range, we have to introduce a scale transformation. Remember, the scale [0, 255] that was defined to mark colors with signs, was arbitrary. We are free to mark the range of colors with other signs (we could use, for example, a [0, 1] scale and to divide it into arbitrary sections). Therefore we propose to shift the scale by -128 to the range [-128, 127]. (Note, that the color sign "0" does not coincide with the cardinal number 0, rather it denotes the section [0, 1) in the number axis, and similarly, "255" denotes [255, 256), and "-128" denotes [-128, -127). So the range of signs [-128, 127) is symmetric with respect to the sign "0", neglecting the openness of the range at the right end.) According to our freedom to denote colors by any sign, this range is an equal right signature to the individual color scales. Nevertheless, this choice provides us with a coincidence in the range of interpretation and the range of values for C, C^S and C^A at each axis (r, g, and b). In other words, values of neither C^S nor C^A go out of the range [-128, 127] where the signs attributed to the color scale are interpreted.

Now, we can assign C^S to each pixel in P^S, so that P^S will be symmetric in its colors, and we can assign a C^A to each pixel in P^A, so that P^A will be antisymmetric in its colors. In this way, we can decompose any, asymmetric picture in a symmetric and an antisymmetric picture.

However, these images cannot be represented, because our computers and our printers are not taught to recognize colors marked with signs in the range [-128, 0). To represent the processed pictures we must execute an opposite scale transformation from the range [-128, 127] to the range [0, 255] towards each color axis. Thus all the original (P) and the processed symmetric (P^S) and antisymmetric (P^A) pictures can be represented. Remember that antisymmetry in the colors means that the pixels in mirror positions are complementary colors, that is $\{C^{Ai,j}{}_r, C^{Ai,j}{}_g, C^{Ai,j}{}_b\}$ and $\{(255\text{-}C^{Ai,j}{}_r), (255\text{-}C^{Ai,j}{}_g), (255\text{-}C^{Ai,j}{}_b)\}$ respectively.

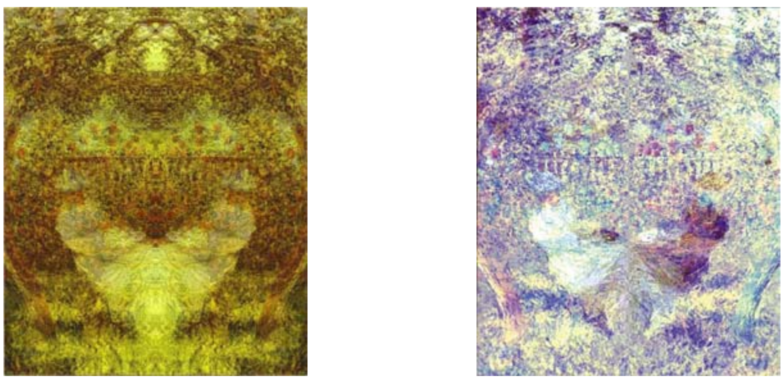


Figure 5: *Symmetric and antisymmetric components of a picture (P^S and P^A).*

Figure 6: *C. Monet:* Alice Hoschedé in the Garden *(1881)*

The preservation of additivity of the operation in mapping the colors into the planar arrangement of pixels guarantees that the sum of color codes (signs) of the individual pixels in $P^S_{i,j}$ and $P^A_{i,j}$ will give the color code (sign) of the original picture's respective $P_{i,j}$ pixel. If one prints the pictures P^S and P^A on transparent folios, and brings the two folios into correspondence, one can reproduce the original picture P as it was before the decomposition. Figure 6 [3] is either the original image (P) before decomposition, or the result of correspondence (addition) of pictures P^S and P^A.

The following examples show the decomposition of the well-known yin-yang symbol into symmetric and antisymmetric components along its four different axes [2] by the application of the above described procedure [1]. If you bring into correspondence the left (symmetric) and right (antisymmetric) images, you get the reconstructed yin-yang shown in Figure 7.

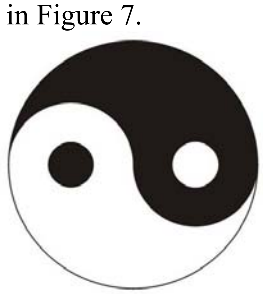


Figure 7: *The rotational antisymmetric yin-yang.*

Figure 8:
symmetric component of a yin-yang *antisymmetric component of a yin-yang*
decomposed along its horizontal *axis*.

Figure 9:
symmetric component of a yin-yang *antisymmetric component of a yin-yang*
decomposed along its vertical *axis.*

Figure 10:
symmetric component of a yin-yang *antisymmetric component of a yin-yang*
decomposed along its main diagonal.

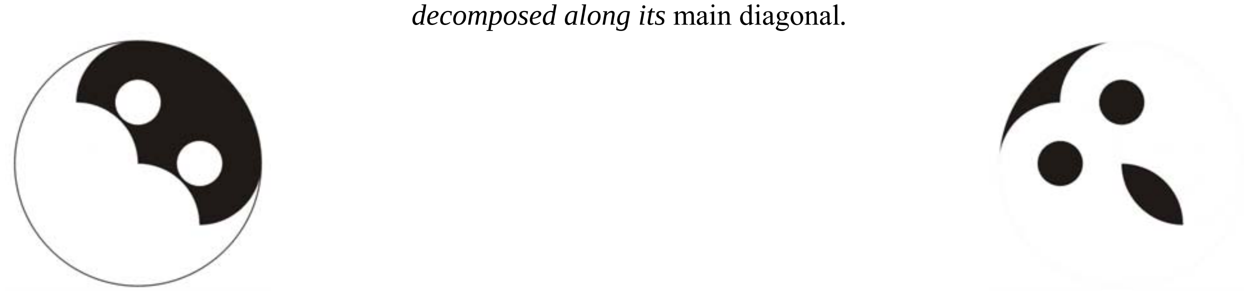


Figure 11:
symmetric component of a yin-yang *antisymmetric component of a yin-yang*
decomposed along its side diagonal.

Figures 8 and 9, as well as 10 and 11 demonstrate another hidden symmetry of the yin-yang, namely when the axis is rotated 90 degrees they produce the same decompositions, swapping the roles of the symmetric and antisymmetric components. Note, that in general, the antisymmetric component contains the information on the deviation from symmetry of the decomposed, original image. This is the basis of the practical technological applications of the method.

We can not only decompose pictures into symmetric and antisymmetric components, but also compose pictures from a symmetric and an antisymmetric image. We know artworks that are painted/printed on two parallel installed transparent plates. They provide different images on looking through them at different angles. This corresponds to the projections treated in paragraphs 1-6. One can also use the inversion of the method described in the rest of the paper above to produce artworks. A symmetric picture painted/printed on 'plate 1' and an antisymmetric image painted/printed on 'plate 2' can produce an asymmetric image. A specific case can be produced when looking at the installation from a perpendicular angle (other angles do not bring all pixels in the two plates into correspondence, cf. the projected artworks by Pankin). This application of the inverse method seems a realistic way of creating geometric artworks.

The original patent application was for technological applications, however the decomposition procedure has a peculiar philosophy: the *world* is – as we see it (for our eyes perceive pixels instead of points in a continuum, cf. paragraph 1), and as it is indeed, with rare exceptions – *asymmetric*, and it is a unity of *symmetry* and *antisymmetry*. Asymmetry = Symmetry + Antisymmetry. Artworks, at least planar ones, demonstrate this unity, using the homomorphism of projection from the abstract 3D space of the color cube into the 2D space (plane).

I presented a kind of mapping in art from real 3D space to a 2D plane and then from an abstract 3D space to a 2D plane. I can only refer to the literature to show that such mapping plays an important role beyond the arts. It also plays a crucial role in science when mapping between multidimensional abstract physical fields and four-dimensional space-time. Its importance is evident in contemporary physical theories, which are based on the mathematical theorems by E. Noether [5] developed further by Utiyama [6],[7], and are called, after H. Weyl [8], gauge theories. The artistic instances treated above are simplified applications of these mathematical theorems, and provide an example of a connection between arts and sciences.

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