# An Origami Puzzle of Intersecting Cubes

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## Abstract

We present a novel origami model similar to David Mitchell's Columbus Cube.
Several pieces can be used to form a put-together puzzle in a variety of ways.
We explore the mathematical aspects of designing the compound model.

## 1. Introduction

The motivation of our investigation is an origami model by internationally known paper folder David Mitchell titled "Columbus Cube" [3] – a cube design with an inverted corner – shown in Figure 1. Using several pieces of this model, compounds of cubes can be formed; examples are the "Ring-Of-Five-Cubes" and "Ball-of-Cubes" models (see [3]). In [2] we determined the exact geometry for these models and for a collection of similar compounds of Platonic polyhedra, which exhibit characteristics of put-together puzzles.
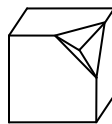


**Figure 1:** *Columbus Cube.*

A cube may be inverted/dimpled at an edge instead of a corner. Several cubes, each inverted at an edge, may be assembled to form a ring (see Figure 2) provided that the geometry of the dimples satisfies some criteria. The dimples may not be uniform, and for us, this is the interesting case, since then the order of the cubes in the ring is not arbitrary and the assembly of the compound will not be a trivial matter. In this paper, we present an origami model for a cube
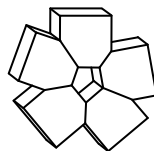


**Figure 2:** *Five cubes forming a ring.*

inverted at one of its edges, as well as an algebraic result that may help the paper folder, using this model, to design a ring of cubes of a particular shape. The ring obtained this way can be viewed as a put-together puzzle with varying levels of difficulty depending on the geometry of the design.

## 2. Origami

In theory, one may consider any compound of intersecting polyhedra and describe its geometry.

For origami models the configuration must be foldable in some reasonable way. One important requirement is that the intersection be symmetric in order for our model to be foldable, that is, the dimple is the mirror image of a portion of the solid about a plane that cuts through the solid. This configuration has a good chance to be foldable unlike a non-symmetric intersection which in most cases is not possible to fold in any simple manner. For the dimpled cubes in this paper we have a simple folding solution. First think about how 3 copies of the 1 x 4 piece in Figure 3 could be
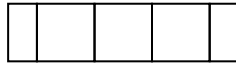


**Figure 3:** *Folding a cube.*

used to form a cube. This design is essentially the same as the "Jackson Cube" where 6 pieces are used, each of size 1 x 2 (the diagram in Figure 3 cut in half at its vertical axis of symmetry), see [3]. In the assembly one would make certain that all three pieces alternate between being "over" and "under". We add creases to form a dimple as shown in Figure 4 (with solid lines for valley folds, and dotted lines for mountain folds). Note that the angle and depth at which the cube is dimpled can be controlled by appropriately placing these additional creases. The assembly of the dimpled cube may be challenging at first; some readers might find it easier to use a version based on the Jackson Cube (cut each piece in half in Figure 4).



**Figure 4:** *Folding a cube with a dimple at an edge.*

Figure 5 shows a folded and assembled model. The dimple has two rectangular and two triangular sides providing sufficient support for the cube that is inserted here to stay in place – no glue or tape is needed. Note that this cube may be positioned in two different ways as a member of a ring. In a design, one might have assumed that it is face *F* that will be on the inside surface of the ring, however, it is possible to turn the cube in such a way that face *G* will end up there.
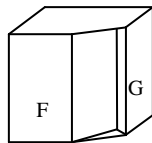


**Figure 5:** *A cube inverted at an edge.*

This fact may be used to increase the challenge for anyone trying to assemble the ring since it is unlikely that using some of the cubes in the "wrong" way will lead to a perfect fit. At the same time, it may be better to prevent this kind of confusion; and we can do so by using three different colors for the three strips of paper in Figure 4, so that the inside of the ring would be required to have a particular color.

### 3. Shapes and Sizes

The geometry of the ring of cubes that we are interested in can fully be described in two dimensions. A square will represent each cube and adjacent squares will have a symmetric intersection as shown in Figure 6. The symmetry requirement for the intersections translates to having equal distances, shown as $x_i$ in the diagram, for each pair of adjacent cubes. We ask the following question: *For what values of $a_1, a_2, ..., a_n$ do rings of cubes exist?*

This may be of interest to an origamist who wants to control the shape of the ring of cubes. We will assume that the $a_i$ satisfy the generalized triangle inequality so that these distances can be the sides of a polygon. For a given set of the distances $a_i$ there are $n$ - 3 degrees of freedom that may be used for varying $n$ - 3 of the angles of the polygon. For an actual model we would set these angles which together with the $x_i$ would define the dimples and the shape of the ring.
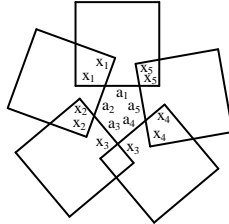


**Figure 6:** *Squares representing cubes.*

Assuming a unit edge length for the cubes (squares), if a ring of squares can be formed, then the $x_i$ satisfy the following system of linear equations:

$$
\begin{aligned}
x_1 \quad + \qquad\qquad\qquad\qquad\quad x_n &= 1 - a_1 \\
x_1 \quad + \quad x_2 \qquad\qquad\qquad\qquad &= 1 - a_2 \\
x_2 \quad + \quad x_3 \qquad\qquad &= 1 - a_3, \\
\quad . \quad\quad . \quad\quad . \quad\quad . \quad\quad . \quad\quad . \quad\quad . \\
x_{n-1} \quad + \quad x_n &= 1 - a_n
\end{aligned}
$$

Surprisingly, parity of $n$ makes a difference. If $n$ is odd, the rank of the coefficient matrix is $n$, and if $n$ is even, the rank is $n - 1$. Thus, there is always a unique solution for $n$ odd, and there are infinitely many solutions for $n$ even provided that the $a_i$ satisfy the equation

$$
a_1 + a_3 + ... + a_{n-1} = a_2 + a_4 + ... + a_n.
$$

To check these statements, successively subtract row $i$ from row $i + 1$ for $i = 1, 2, \dots, n - 1$ in the augmented matrix. Then, the solutions are given by

$$
x_i = \frac{1}{2} - \frac{1}{2} \sum_{j=1}^{n} (-1)^{i-j \,(\mathrm{mod}\, n)} a_j \quad (i = 1, 2, ..., n),
$$

if $n$ is odd, and

$$
x_n = \text{arbitrary};
$$

$$
x_i = \frac{1}{2} + (-1)^{i-1} \frac{1}{2} + (-1)^{i} x_n + \sum_{j=1}^{i} (-1)^{i+j-1 \,(\mathrm{mod}\, n)} a_j \quad (i = 1, 2, ..., n-1),
$$

if $n$ is even. It is worth noting that the matrix of the system above is a *circulant* matrix and systems of linear equations with such left-hand sides can be solved using the Discrete Fourier Transform (see P. J. Davis [1]). We require all $x_i$ to be nonnegative. By reducing all $a_i$ by the same percentage, we can always achieve this, while preserving the relative sizes of $a_i$ (which may be important for a design with a particular shape). However, even with all $x_i$ positive, it may not be possible to fold every dimpled square. No matter how small (or large) the $a_i$ are, it may happen that for some of the squares (cubes) *two* corners (edges) would be forced to participate in the dimple – call this a non-simple intersection –, for which we do not have a simple folding

solution. Figure 7 shows the case of $n = 3$ with the inside of the ring forming an isosceles triangle $(a_1 = a, a_2 = a_3 = b)$. The reader can verify that if $\beta \leq \arctan \dfrac{4 - \sqrt{7}}{6} \cong 12.72°$, then the two squares will have a non-simple intersection regardless of the actual values of $a$ and $b$.
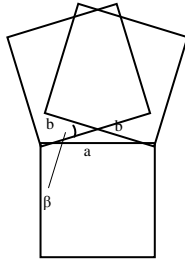


**Figure 7:** *Non-simple intersection.*

One could use rectangles (squares stretched in the radial direction) to find a way around this problem; the folding diagrams for the corresponding stretched cubes would require only minor changes relative to Figure 4.

## 4. Conclusion

Results presented in this paper are helpful in understanding and designing the details for a family of put-together origami puzzles. These puzzles are three-dimensional models made of cubes, however, the essential part of their geometry can fully be described in two dimensions. Hence, there is a two-dimensional version where the cubes are replaced by squares and the resulting puzzles may be viewed as a type of tangram. The 3D origami models may offer a more satisfying experience, and they can also be displayed as decorative items.

It may be possible in some cases that the arrangement of the cubes in an order that is different from what we had in the original design still works; the cubes may fit perfectly together to form a ring in more than one way. Thus, it would be useful to find a description of how the geometry of the ring $R = (A_1, A_2, \dots, A_n)$ affects the set of permutations

$$S(R) = \{p \in S_n \mid (A_{p(1)}, A_{p(2)}, \dots, A_{p(n)}) \text{ form a ring in this order}\}.$$

For an actual model (2D or 3D) we need to have some control over $S(R)$. There should not be too many or too few possibilities (permutations) to try. For $n$ (= the number of pieces) as small as 4, we can design puzzles that are sufficiently challenging with a 1 in 6 chance for success. The number of possibilities, $(n-1)!$, grows quickly with $n$, but if $S(R)$ is also large, we may have an entertaining rather than frustrating puzzle. We encourage the reader to design his or her own ring-of-cubes puzzle.

### References

[1] Davis, P. J., *Circulant Matrices*, 2<sup>nd</sup> ed., New York: Chelsea, 1994.
[2] Lengvarszky, Z., Compound Platonic Polyhedra in Origami, *Math Mag* (79:3) 186-194, 2006.
[3] Mitchell, D., *Mathematical Origami*, 2<sup>nd</sup> Edition, Tarquin Publications, 2002.