Regular Polyhedral Lattices of Genus 2: 11 Platonic Equivalents?

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Abstract

The paper observes Euler's formula for genus 2 regular polyhedral lattices is obeyed by *at most* 11 cases of the Schläfli symbol {p,q}, where p is the number of edges of each face and q the number of faces meeting at each vertex. *At least* one example is given for the 'first' 6 cases, but not for their 5 'duals'. The examples are known from various sources, but their present classification suggests they are lookalikes of classical Platonic equivalents. An 'artistic' corollary is the observation that hyperbolic geometry models can be constructed using Zometool.

Euler's Formula on a Spreadsheet

Euler's (-Poincaré's) formula for polyhedra states that

$$V + F - E = 2 - 2g \qquad (1),$$

where V stands for the number of vertices, F for the number of faces, E for the number of edges and g for the genus. The Schläfli symbol {p,q} specifies the number of edges p of each face and the number of faces q meeting at each vertex. Since any edge joins two vertices and has two adjacent faces, pF = 2E = qV. The substitution of E = pF/2 and V = pF/q in Euler's formula implies the following expressions for V, F and E:

$$V = 4p(g-1)/(pq-2p-2q),\ F = 4q(g-1)/(pq-2p-2q),\ E = 2pq(g-1)/(pq-2p-2q) \qquad (2).$$

There are five 'Platonic solids', as common arguments going back to Euclid's Elements show (see [5] for a modern interpretation). A 'topological proof' is based on the formula (1) for g=0. It implies

$$1/p + 1/q = 1/2 + 1/E > 1/2,$$

and this is only possible for (p, q) equal to (3, 3), (3, 4), (3, 5), (4, 3) or (5, 3). Note this does not complete the proof (as sometimes stated), since a convexity condition should be added to ensure uniqueness.

A less elegant yet more 'modern' argument simply lists all possibilities for V, F and E based on formula (2) for g=0, and lets a computer select all acceptable cases where V, F and E are positive integers greater than 3, for instance through a simple spreadsheet program. Of course, this does not construct a proof, in a mathematical sense, as the search process never really finishes (see table 1).

	p	q	V	F	E	Name
1)	3	5	12	20	30	icosahedron
2)	3	4	6	8	12	octahedron
3)	3	3	4	4	6	tetrahedron, dual of itself
4)	4	3	8	6	12	cube, dual of (2)
5)	5	3	20	12	30	dodecahedron, dual of (1)

Table 1: *Output of (2) for g=0.*

The above listing provides an easy procedure: it is enough to make a spreadsheet listing, and no reasoning is required. It was given as an introduction to the next situation. Indeed, one can proceed similarly for g=2 in formula (1). It now implies

$$1/p + 1/q = 1/2 - 1/E < 1/2.$$

This is not a lower bound, as in the case of the Platonic solids, but an upper limit. Thus, the above argument does not seem to lead anywhere, as the integers p and q increase indefinitely and 1/p + 1/q becomes smaller. Nevertheless, a similar simple spreadsheet output shows only 11 cases emerge because all other possibilities do not yield acceptable positive integer solutions for V, F and E (see table 2).

	p	q	V	F	E	Name
1)	3	7	12	28	42	icosahedron + 2 open anti-prism ‘tunnels’
2)	3	8	6	16	24	octahedron + 2 open anti-prism ‘tunnels’
3)	4	5	8	10	20	cube + 2 open cube ‘tunnels’
4)	3	9	4	12	18	tetrahedron + 2 open anti- prisms ‘tunnels’
5)	4	6	4	6	12	cube + 1 open cube ‘tunnel’
6)	5	5	4	4	10	open pentagonal solid, ‘dual’ of itself
7)	6	4	6	4	12	‘dual’ of (5)
8)	9	3	12	4	18	‘dual’ of (4)
9)	5	4	10	8	20	‘dual’ of (3)
10)	8	3	16	6	24	‘dual’ of (2)
11)	7	3	28	12	42	‘dual’ of (1)

Table 2: *Output of (2) for g=2.*

As the table showed there are *at most* 11 cases, we now try to provide *at least one* example for each case. We will explain the proposed names for the cases 1- 6. The genus of a lattice will be determined algebraically, by ‘checking out the numbers’, given in the formulas.

Examples of Infinite Polyhedra for the First 6 Cases

1) Case p = 3, q = 7, V=12, F=28, E=42. The best way to learn about polyhedra is to make models (says [2]), and so we start building an icosahedron with Zometool (see [11]). Now a triangular anti-prism can be put on two of its faces. Such a polyhedron is composed of two parallel copies of an equilateral triangle, connected by an alternating band of equilateral triangles, and is, in fact, an octagon. The two triangular anti-prisms are ‘open’, as they will be used as ‘tunnels’ to connect them to other icosahedrons.

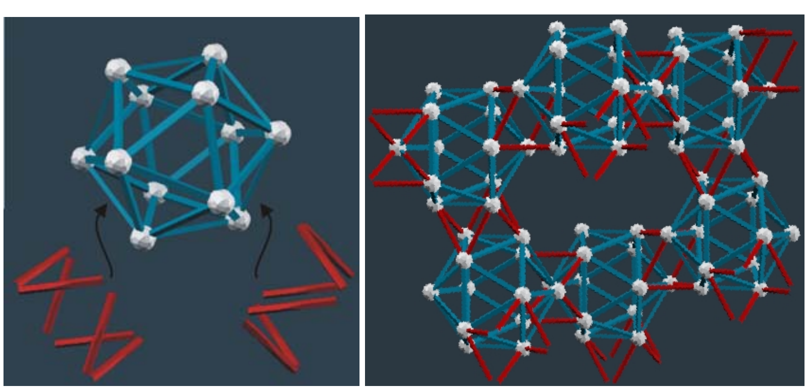

Figure 1: *Forming an ‘atom’ with its ‘connectors’ (a), and a loop formed by six elements (b).*

Thus, the basic element consists of 1 icosahedron and 2 open anti-prisms (note there are 2 possibilities: 2 'mirror images' or 'enantiomorphous' forms). Several of these basic elements together form loops and complete a larger structure, giving an impression of the infinite polyhedron thus created. Since each atom has 4 'arms' going off in tetrahedral directions, the most natural and most symmetrical way of putting these minimal repeatable units together is by forming a 'diamond lattice.' On the Zome model it is easy to see 12 vertices suffice to put the model together. The open icosahedron has 20 faces, but on 2 of them anti-prisms were set, while 2 others do not count as they are used to connect them to other elements. This makes 16 faces, plus twice 6 for the anti-prisms, a total of 28. The icosahedron has 30 edges, of which 2 times 6 edges for the open anti-prisms have to be added, thus E=42. Thus, formula (2) fits.

2) Case p = 3, q = 8, V=6, F=16, E=24. As the number of edges diminishes (and as the symmetry properties change), we abandon Zome and opt for virtual models. The 'atom' with which we start is an octahedron, of which four that do not share any edges are removed. We again put 6-sided open triangular anti-prisms on it. Several of these elements together allow eight equilateral triangles to meet at each vertex, forming an infinite solid with the structure of a diamond lattice.

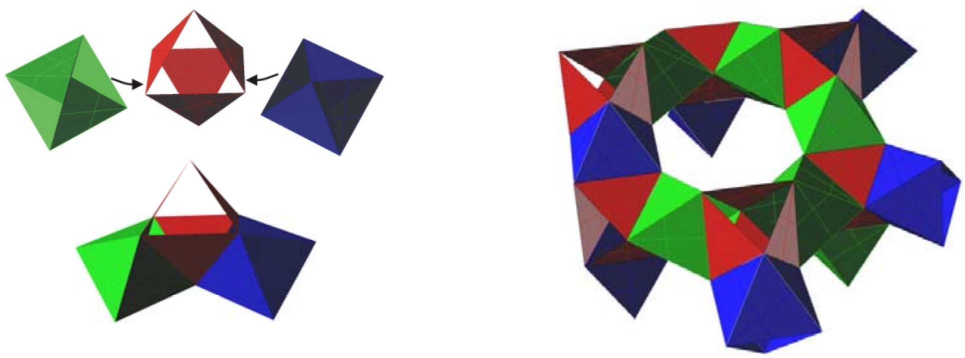

Figure 2: *Forming the 'atom' with its 'connectors' (a, b), and a general view (c).*

The middle open octahedron has 6 vertices, and none are added by the other triangular anti-prisms, since they are open solids. Both have 6 faces, while the middle open triangular anti-prism adds only 4. They twice have 12 edges, of which the 2 times 3 edges of the open faces have to be removed, and 8 edges of the middle open triangular anti-prism have to be added.

3) Case p = 4, q = 5, V=8, F=10, E=20. Now the atom is a cube, of which only two perpendicular faces remain. Two open cubes are attached as connectors. Again, two faces are removed from the middle cube. Putting several of these elements together, five squares meet at each vertex (p=4, q=5).

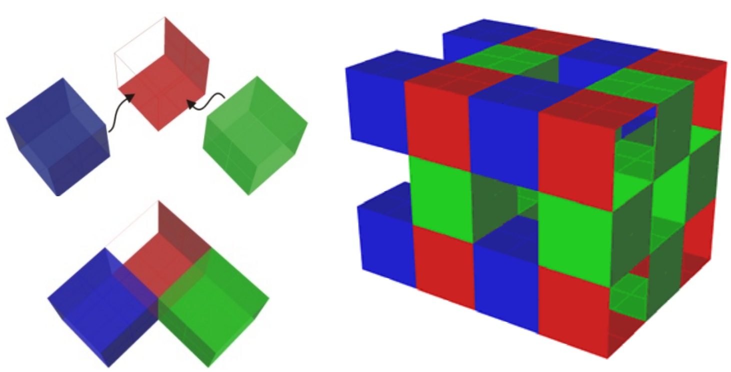

Figure 3: *Forming the 'atom' with its 'connectors' (a, b), and a general view (c).*

The middle open cube has 8 vertices, and none are added by the other cubes, since they are open solids. There are 2 + 4 + 4 faces, a total of 10. The middle cube has 12 edges, to which 2 times 4 from the open cubes put on it, have to be added; thus E=20.

4) Case p = 3, q = 9, V=4, F=12, E=18. In this case the atom is a tetrahedron, of which first two opposed faces are removed. On the open faces, open octahedrons are placed, that is, 6-sided anti-prism connectors formed by equilateral triangles. The two remaining faces of the middle open tetrahedron are now removed, so that in fact only the edges of the tetrahedron remain. Putting several of these elements together, the result is a 'square' lattice, and 9 equilateral triangles meet at a vertex (p=3, q=9).

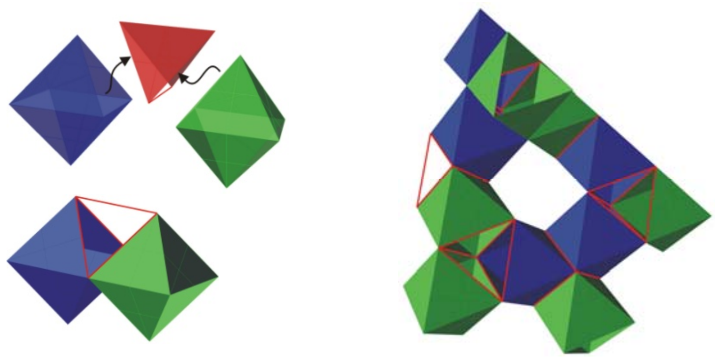

Figure 4: *Forming the 'atom' with 'connectors' (a, b), and a grouping of some elements (c).*

The middle tetrahedron has 4 vertices, and none are added by the octahedrons, since they are open solids. Both octahedrons have 6 faces, while the tetrahedron adds none. They twice have 12 edges, of which the 2 times 3 edges of the open faces have to be removed.

5) Case p = 4, q = 6, V=4, F=6, E=12. Here, an example will be used suggested by Chaim Goodman-Strauss and John Sullivan (see [3]). The atom is a cube, of which two opposite faces are removed while 2 squares serve as connectors. The latter look as the opposite sides left over from another open cube. Or, alternatively, and closer to the previous systematic description paradigm, the 2 parallel square plates can be seen as the atom, and the open cube (a cube of which 2 parallel faces were removed) as connector. Note each subsequent layer is turned 90° compared to the previous layer.

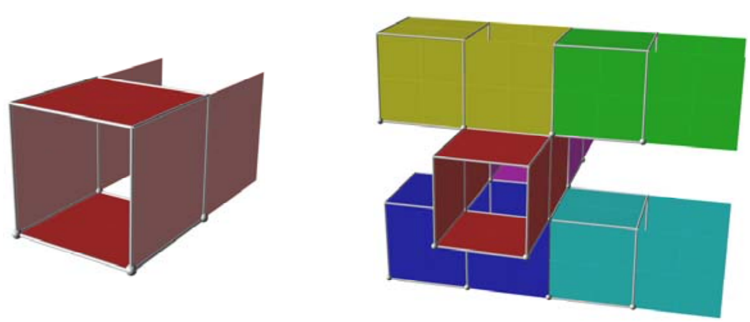

Figure 5: *The basic element (a) and a general view (b).*

To make the arithmetic easier, the 4 vertices that are still counted in each atom were indicated as small spheres (the lower vertices in figure 5 a, b). Similarly, the 12 edges that are counted were drawn as more clearly in figure 5. When the atoms are put together, it turns out all vertices and edges are counted once.

6) Case p = 5, q = 5, V=4, F=4, E=10. Here, the atom is an open solid formed by four pentagons (figure 6 a). First, they are put in rows such that 1 edge is common, and 2 other edges are parallel (figure 6 b).

Perpendicular to this row, other rows complete the lattice (figure 6c) and now, five pentagons meet at each vertex (p=5, q=5).

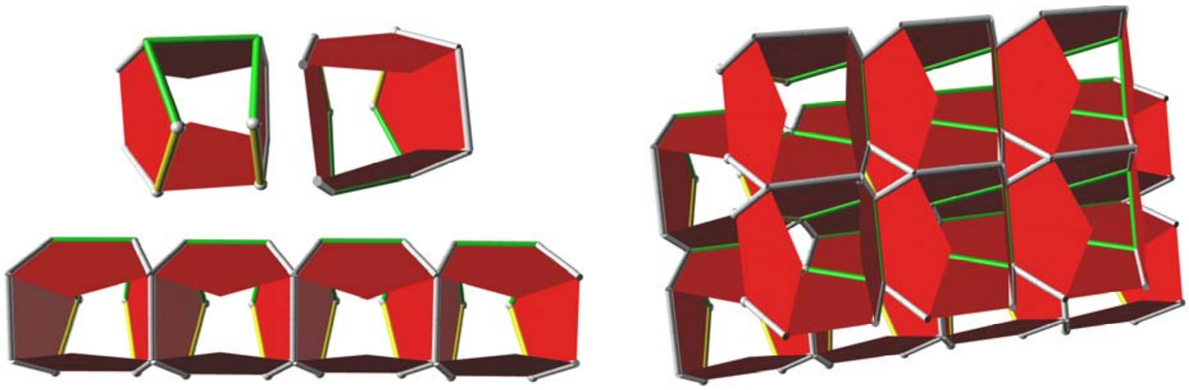

Figure 6: *Views on the basic element (a), a row of those atoms (b), and a view on the final lattice (c).*

Again to facilitate the arithmetic, the 4 vertices that are counted in each atom were indicated as small spheres, while the edges that are counted were drawn as tiny cylinders in figure 6. In each atom, there are 4 faces, while of the total of 16 edges, only 10 are taken into account. It is harder to see it is enough to count but 4 of the 12 vertices of the pentagonal atom, yet 5 pentagons meet in each point and so the numbers do fit.

Notes

Duality. For the cases 6 to 11, we could think of constructing 'duals' of the proposed infinite solids, by connecting the midpoints of each face. It would create solids with faces having as many vertices as there were originally faces meeting in a vertex. However, these faces are not necessarily flat anymore, as was the case for the classical Platonic solids. For instance, the pentagonal case 6 would be its own self-dual but this works only very approximately (see figure 7). Thus, the word 'dual' was put between apostrophes. Referees of the present paper pointed out it would be interesting to find out for which cases the simple geometrical mid-point construction actually works.

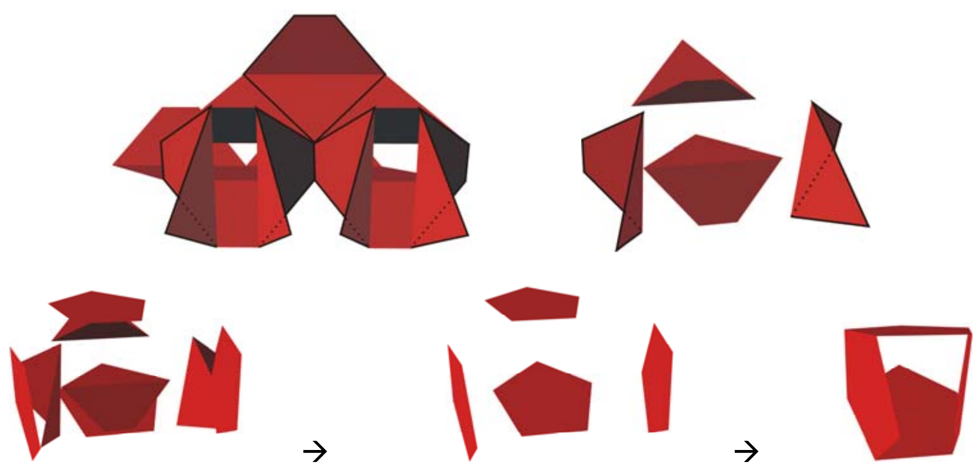

Figure 7: *'Dual' of figure 6 (a) and a detail of 4 'dual' curved pentagons (b). When they are 'flattened' (c), one recognizes (d) the atom of figure 6 (e), so that the lattice might be seen as 'self-dual'.*

Uniqueness. The provided examples are not necessarily unique. Coxeter proposed a modified Schläfli symbol with an additional parameter to differentiate between different possibilities. For case 3 for instance, it is straightforward to propose an example that looks different, as the assembled infinite polyhedron is completely flat though the elementary polyhedron is but slightly different. Yet, the count in the formula $V - E + F = 2 - 2g$ remains valid as $V=8$, $E=20$ and $F=10$ and $g=2$. Note the uniqueness of the Platonic solids needs more than Euler's identity too, as a convexity condition has to be added.

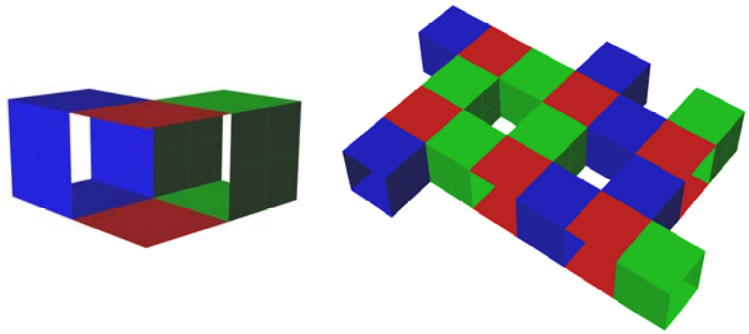

Figure 8: *Another example for the case p=4, q=5.*

Rejections. Coxeter disqualified some examples as (infinite) 'regular polyhedra' because they have adjacent coplanar faces. This would mean only cases 1 and 2 are acceptable. This condition probably is too strict, but an 'extreme' case most readers will surely want to reject is an alternative example for case 5: a set of cubes stacked on the corners of each other. Of a cube's 8 vertices, 4 were removed, as they would otherwise be counted twice, so that the model 'fits the numbers'. Indeed, 6 squares now meet at each vertex, but the corner points of the cubes can hardly be seen as 'connectors'.

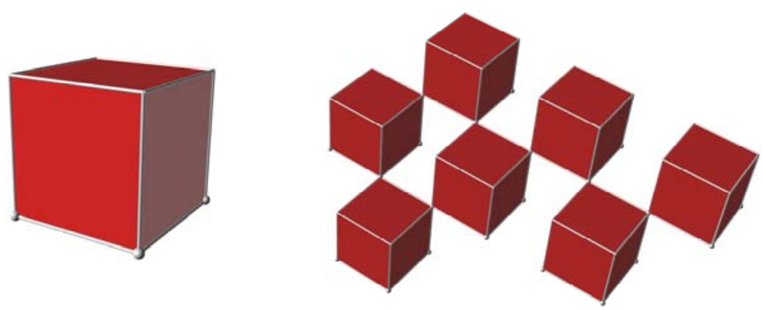

Figure 9: *The basic element with four vertices to be counted (a), and a general view (b).*

This example illustrates the need for a well-defined description of a 'Platonic' infinite lattice of genus 2. In the genus 0 case, there is a neat difference between the classical Platonic and the Kepler-Poinsot solids, though they both obey the initial formula (2). A similar approach for the genus 2 situation is desirable.

Arithmetic vs. Geometry. The present paper followed an 'arithmetical' approach: based merely on formula (2), it was concluded that only a limited number of regular polyhedral lattices of genus 2 exist, and the properties of the given examples were verified by 'checking out the numbers'. The procedure was followed throughout, and it may have been disappointing for the more geometrically inspired reader. A more intuitive reference on the genus of the given infinite regular lattices, which obviously have infinitely many tunnels and handles, is given in [3]. It can be easily used to explain the constructions of cases 1 to 5: each time, they start with a Platonic solid connected by 4 shared tunnels to the other elements, thus creating 2 handles 'for each atom' (and thus explaining the genus is indeed 2). Case 6 however raises more questions: the lattice does not even seem to have an 'inside' or an 'outside'. Fortunately, as a part of his referee report, Carlo Séquin could show it does (see figure 9).

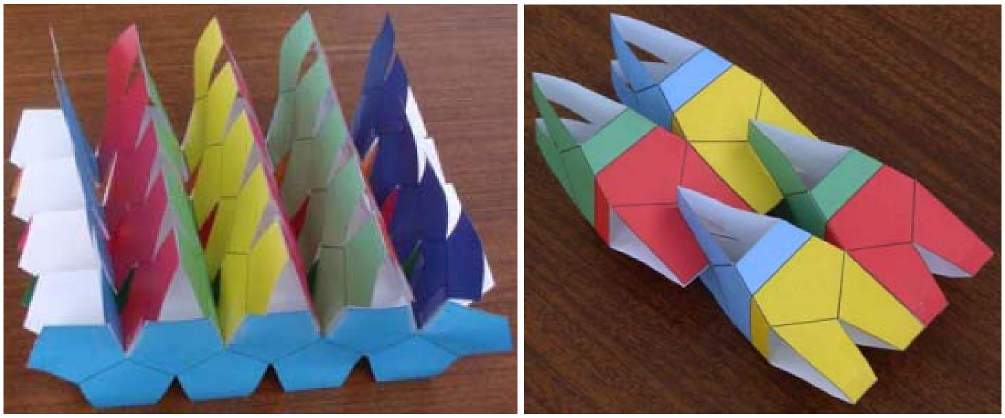

Figure 10: *Carlo Séquin's paper model for case 6.*

Séquin looked for a 'true solid', which can be repeated as a unit with exactly 4 openings, through which it connects to its neighbors. To make it symmetrical and repeatable, and clearly see the four openings, he had no other solution but to cut all pentagons into two parts. Yet, it surely helps the geometrical readers, to accept this case as a true genus 2 example, in case they would be not have been convinced by the arithmetic counting process.

Application

The examples can be qualified as 'hyperbolic' in the sense that the angular excess of polygons meeting in a vertex exceeds 360° (see [1], [9]). Of course, they are far from simply connected, while a truly hyperbolic plane should be simply connected. Nevertheless, the first example brings to mind 'hyperbolic geometry models' can be constructed using the Zome tool, since it shows 7 equilateral triangles meeting at a vertex. This was a hyperbolic feature proposed by Thurston (see [9]). Diana Taimina is known for her crocheted models of the hyperbolic plane (see [7], [8]), but as few mathematicians are experienced crocheters, Zome can be a good alternative for constructing models similar to Taimina's. Constructing a 'giant hyperbolic disk', curving away from itself at every point, would allow making hyperbolic geometry drawings (see [1]), if panels between the Zome struts can be added as in vZome (see figure 12).

Figure 11: *Taimina's 'Crocheting Adventures' (a) and similar hyperbolic adventures by architecture students Nele Demartin and Sarah Schouppe (Sint-Lucas Brussels, Belgium).*

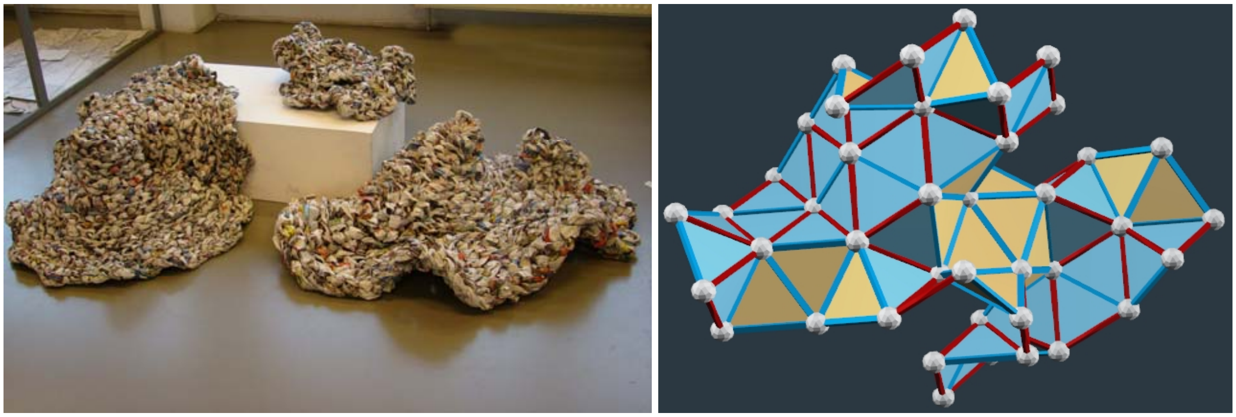

Figure 12: *More hyperbolic plane models in knotted newspaper strings (a) and Zome approximation of the hyperbolic plane, with additional panels between the vertices (b).*

The above examples were a collateral encouragement of students' work in the field, but, as some referees pointed out, there surely are more applications of the present topic (see [4]). Furthermore, it remains to be seen what other 'hyperbolic' examples and regular polyhedral lattices of genus 2 can be made through Zome. Finally, note a similar approach for a higher genus seems promising (see [6]).

Acknowledgements

The author thanks Scott Vortmann for his help with vZome. Referee Carlo Séquin did not only raise the right questions, confirmed by other referees, he also solved some of them.

References

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