Branching Miter Joints: Principles and Artwork

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Abstract

A miter joint connects two beams, typically of the same cross section, at an angle such that the longitudinal beam edges continue across the joint. When more than two beams meet in one point, like in a tree, we call this a branching joint. In a branching miter joint, the beams' longitudinal edges match up properly. We survey some principles of branching miter joints. In particular, we treat joints where three beams with identical cross sections meet. These ternary miter joints can be used to construct various branching structures. We present two works of art that involve branching miter joints.

1 Introduction

The miter joint is a well-known way of connecting two beams of the same cross section at an angle. For a regular[1] miter joint, both beams are beveled at half the joint angle. In a properly executed miter joint, the longitudinal edges continue nicely across the joint (see Figure 1). Whatever the first beam's rotation about its center line and whatever the intended joint angle, it is possible to choose the second beam's longitudinal rotation to make a regular miter joint. Thus, there are two independent, continuous degrees of freedom (you can experience this with [2]). The shape of the cross section is also completely free.

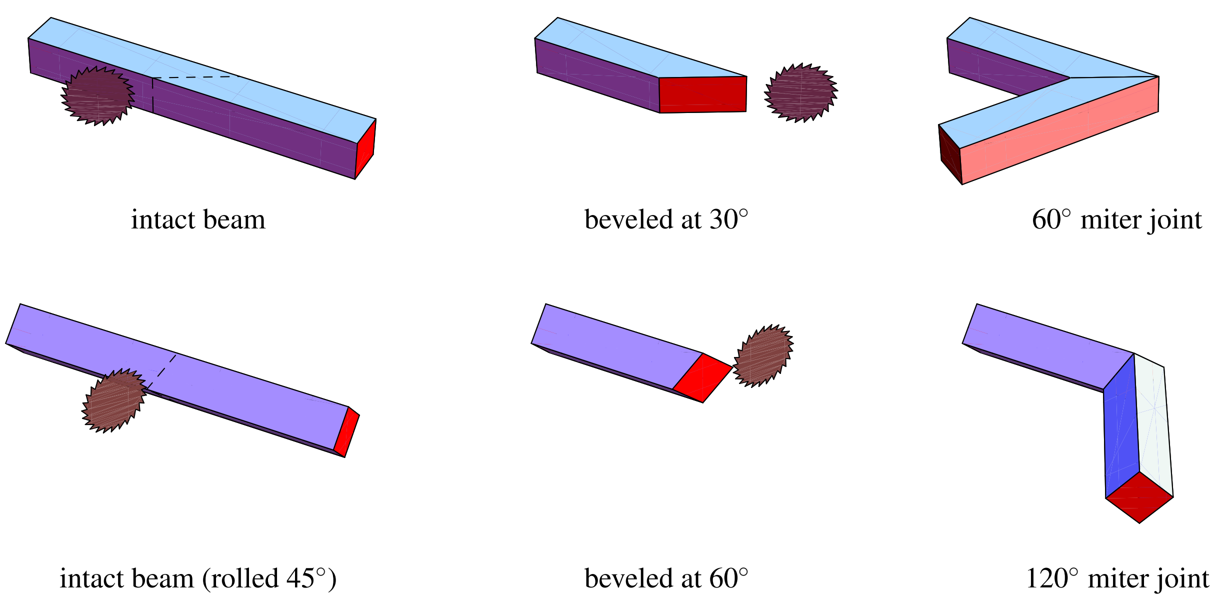

Figure 1: Regular miter joints, showing two independent degrees of freedom: joint angle and roll angle

[1]In a *regular* miter joint, the cut face lies in the internal bisector plane of the joint angle, in contrast to a *skew* miter joint [1].

Miter joints can be used to construct linear figures from beams. Closing such a figure into a loop can be a challenge, since the longitudinal edges can fail to continue across the last joint [1]. In this article, we study another challenge, which arises when connecting more than two beams in the same joint. The goal is again to have longitudinal beam edges match nicely at the joint. We call them *branching miter joints*. These joints are useful to construct more complicated objects, like stick polyhedra and trees.

Section 2 presents some general principles of regular branching miter joints, in particular where three beams of the same cross section meet. We describe the impossible cuboid in Section 3. It involves eight regular ternary miter joints. An object with five joints that each connect four beams is shown in Section 4. Section 5 concludes the article.

2 General Principles for Ternary Miter Joints

Let us first investigate the regular ternary miter joint connecting three beams having the same cross section. Three given line segments, labeled A, B, and C, meet in one point at fixed angles. We will describe the direction of a line segment by its longitude and latitude. In our first examples, we have segment A at 0° longitude and 0° latitude (on the equator); segment B at 90° longitude West and 0° latitude; segment C at 45° longitude West, and 61° latitude North. Consequently, we have $\angle AB = 90^\circ$, the angle $\angle AC = \angle BC \approx 70^\circ$.

These line segments will be the center lines of three square beams to be connected by a branching miter joint. In order to have the longitudinal beam edges meet up properly, each pair of beams needs to form a proper binary miter joint when ignoring the other beam. There are three such pairs.

We start with a square beam that has segment A as center line. It can still be freely rotated, but any rotation of beam A will enforce a rotation of the beam along segment B, and similarly of beam C. However, also the rotation of beam B will enforce a rotation of beam C. Thus, we may have two conflicting requirements on beam C as illustrated in Figure 2.

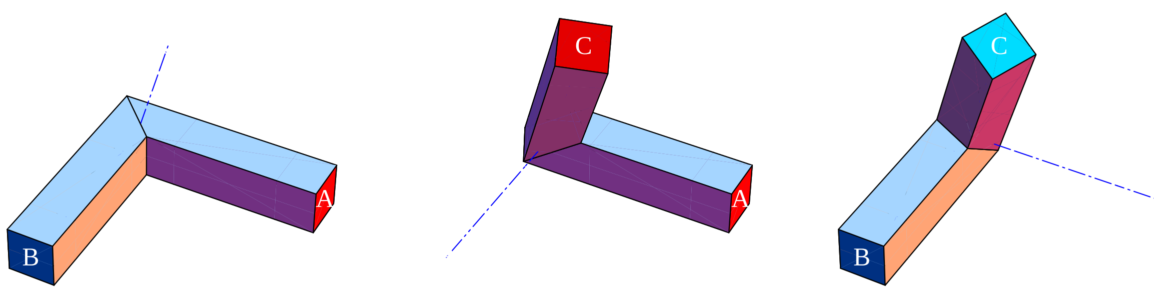

Figure 2: Binary miter joints derived from ternary meeting point: A forces B, A forces C, B forces C

This conflict is even more obvious if we superimpose the two rotations for beam C as enforced by beam A and by beam B. Figure 3 shows the result.

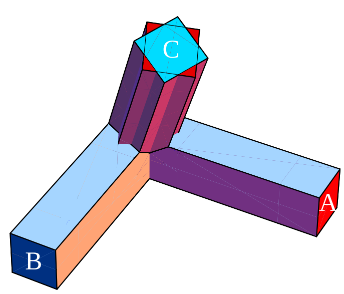

Figure 3: Superimposed binary miter joints derived from ternary meeting point, illustrating a mismatch

If we rotate beam A clockwise, then this enforces *counterclockwise* rotations of beams B and C. But this rotation of beam B will enforce its own *clockwise* rotation on beam C. Therefore, by suitably rotating beam A, the mismatch between the two rotations imposed on beam C can be reconciled, since they rotate in opposite directions. It turns out that there are two qualitatively different ways in which the miter joint can be made to work. On the left in Figure 4, there is a point where three longitudinal beam edges meet (in

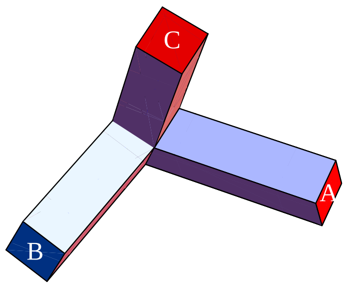

Figure 4: The two ways in which a ternary miter joint can match properly

fact, two such points, but the other one is not visible). On the right in Figure 4, there are no such points; the longitudinal beam edges meet pairwise. There is a ternary meeting point (in fact, there are two such points), but it is a meeting of cut edges.

In summary, if three segments meet in a point with given relationships (the mutual angles), then there is only a discrete set of longitudinal rotations that works. To make a regular ternary miter joint work, you cannot independently vary both the angles and the rotations. One will restrict the other. Alternatively, if we start with a binary miter joint to connect beams A and B, then there is only a limited set of directions from which a third beam C can join this pair and form a proper ternary miter joint.

Figures 5 and 6 concern a pair of square beams A and B mitered at 90° and lying in the horizontal plane. Figure 5 shows the five ways in which a third square beam C can be join them when restricted to the upper half of the bisector plane between beams A and B.

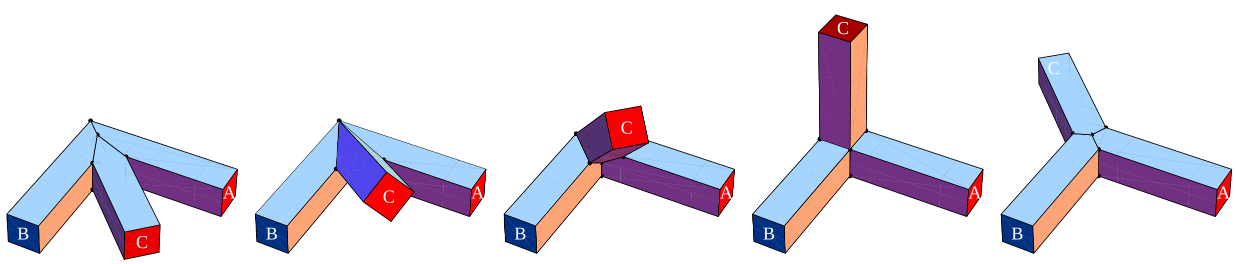

Figure 5: Given a binary miter joint connecting square beams A and B, there are five directions for beam C to make a proper ternary miter joint, if it is restricted to the upper-half of the angle bisector plane

Figure 6 shows from which directions, in general, beam C can meet the mitered pair to make a proper ternary miter joint, i.e., when beam C is not restricted to the bisector plane. The end of beam C away from the joint moves on a sphere. The thick lines correspond to directions where the rotational mismatch (i.e., difference in rotation of the cross sections of beam C as induced by beams A and B) is a multiple of 90°. On the equator the rotational difference is 0°. Taking the direction of beam C equal to either beam A or beam B produces a singularity.

From this figure, it appears that the locus of end points C, such that the ternary miter joint matches, lies in four discrete planes (one of them the horizontal plane, which contains two 'branches'). All these planes contain the end points A and B of beams A and B. We have not proved this, and offer it as a conjecture.

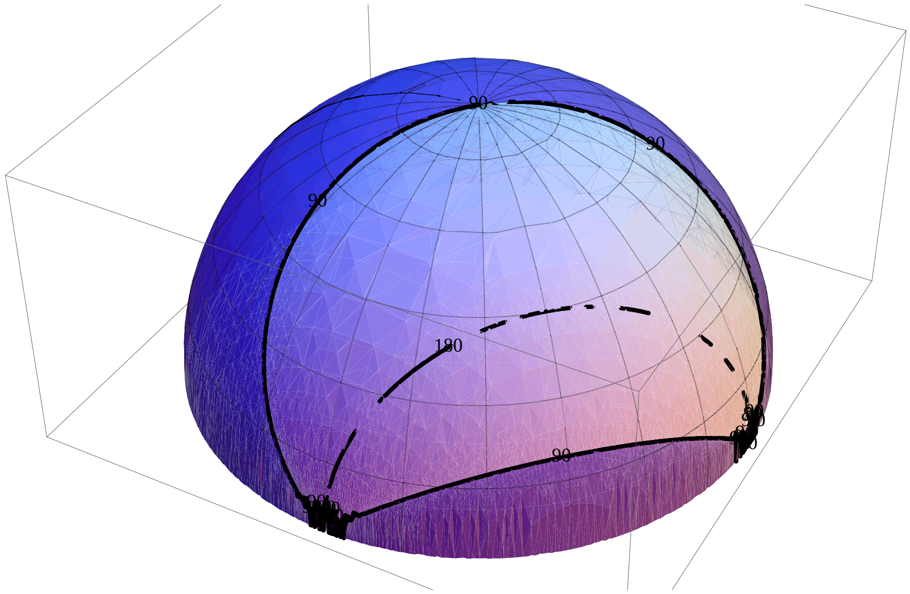

Figure 6: Plot of the rotational mismatch at beam C when square beams A and B are mitered at 90°; the direction of beam C is determined by its endpoint on the sphere; mismatches of 90° and 180° have been marked; on the equator the mismatch is 0°.

In general, the ternary miter joint will match if the amount of rotational mismatch is a symmetry of the beam's cross section. In case of a square beam, this corresponds to multiples of 90°. For equi-triangular beams, the rotational difference needs to be a multiple of 120°.

Note that if the cross section is not mirror symmetric, then a matching ternary miter joint is impossible, because beams A and B induce cross sections at beam C that are mirror images. This is illustrated in Figure 7.

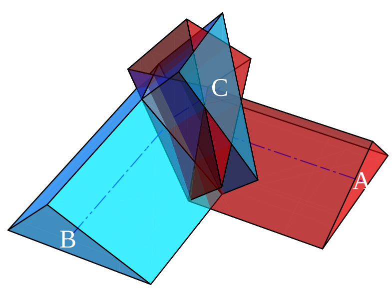

Figure 7: A ternary miter joint is impossible when the cross section is not mirror symmetric.

A different way of understanding all this, is to imagine a closed linear structure following the path C, O, A, O, B, O, C, where O is the central meeting point, and A, B, and C are the endpoints of beams A, B, and C respectively. At A, B, and C we imagine *regular fold joints* [1] at an angle of 180°, and at O there are three regular (binary) miter joints. This closed six-beam figure is self-intersecting; in fact, on OA the two beams coincide, as well as on OB. However, the two beams on OC need not coincide, depending on the rotational difference of their cross section. In [1], this difference is called the *total torsion* along the path. Observe that the number of fold joints involved is *three*, which is an *odd* number. Hence, according to the *Odd Fold Matching Theorem* [1], there are two rotations of the cross section that create a matching joint at C. These two proper closures correspond to the two properly matched ternary miter joints mentioned earlier.

3 The Impossible Cuboid

Next to the stairs in Escher's lithograph *Belvedere*, sits a man holding an impossible cuboid, while looking at the drawing of a Necker cube (Figure 8). This inspired Dick Baas Becking to try and design a real object that would look like the impossible cuboid from an appropriate viewpoint, without resorting to interrupted edges. The Foundation *Ars et Mathesis* commissioned Popke Bakker, an artist known for his use of miter joints, to realize Dick's idea.

Figure 8: An impossible cuboid in Escher's litho *Belvedere* (only a fragment shown)

This design involves eight ternary miter joints. It turned out to be far from obvious how to tune the details of Dick's initial idea so that all ternary miter joints match properly. The second author was then asked to complete the design. He started with two interlocking squares connected by four cross beams. This still leaves a lot of freedom. For each configuration, the total amount of mismatch can be calculated. He wrote a computer program that iteratively searches for a suitable configuration by successively adjusting the parameters to minimize the total mismatch. After some experimentation, the program surprisingly found a feasible solution: see Figures 9 and 10.

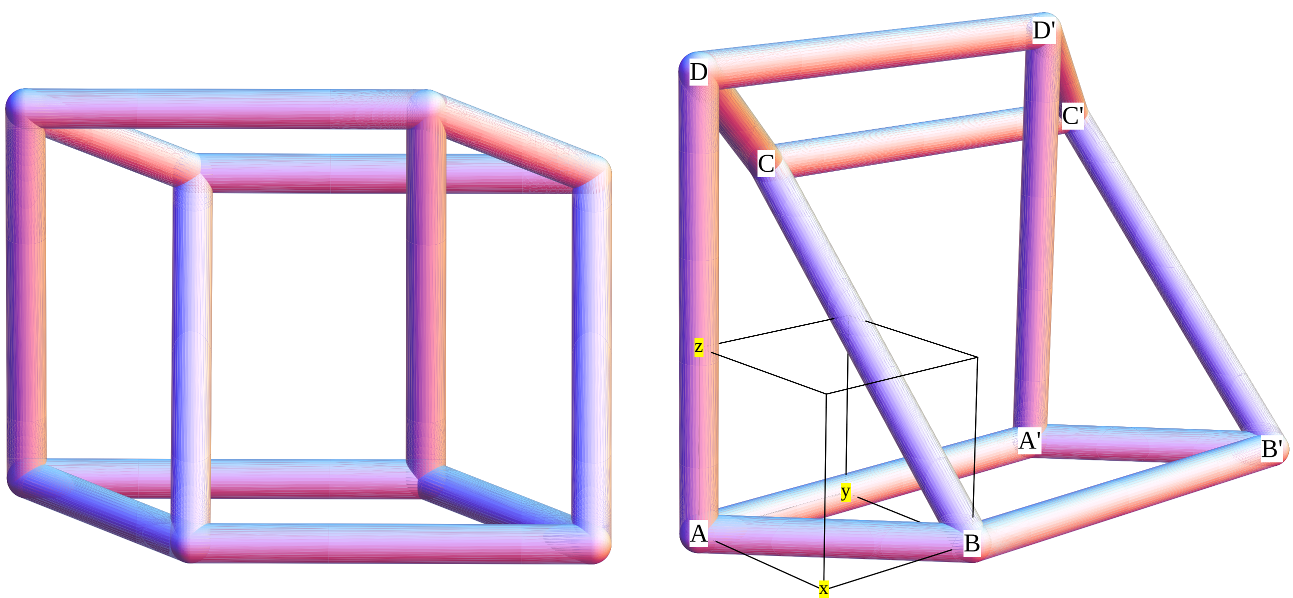

Figure 9: Tubualar version of *The Impossible Cuboid*, viewed from two different angles

The final design shown in Figure 10 consists of twelve square beams connected by eight regular ternary miter joints. Six of these joints are of the first type as shown on the left in Figure 4; the other two have the second type (these two are clearly recognizable toward the center in Figure 10). The design is completely characterized as follows (see Figure 9, left). The six 'faces' of this 'cuboid' are two congruent squares ($AA'D'D$ and $BB'C'C$, with angles of 90°), two congruent parallelograms ($AA'B'B$ and $CC'D'D$ with $\angle BAA' = \angle A'B'B = 45°$ and $\angle AA'B' = \angle B'BA = 135°$), and two congruent non-planar quadrangles ($ABCD$ and $A'B'C'D'$ with $\angle ABC = \angle CDA = 60°$, and $\angle DAB = \angle BCD = 90°$). The beam lengths satisfy $AB : BC = 1 : 1 + 1/\sqrt{2} \approx 7 : 12$. The beams are rotated $\arctan(\sqrt{2} - 1) = 22.5°$. Initially, we were amazed that his design involves only such 'nice' angles. In hindsight it can be understood why this design works, without resorting to computer programs. In fact, there is another solution by rotating the beams over 45°. All ternary miter joints than change their type. Popke Bakker realized the resulting design in various sizes and materials. A large stainless steel version (see Figure 10) is located on the Art Route at Erasmus University Rotterdam [3].

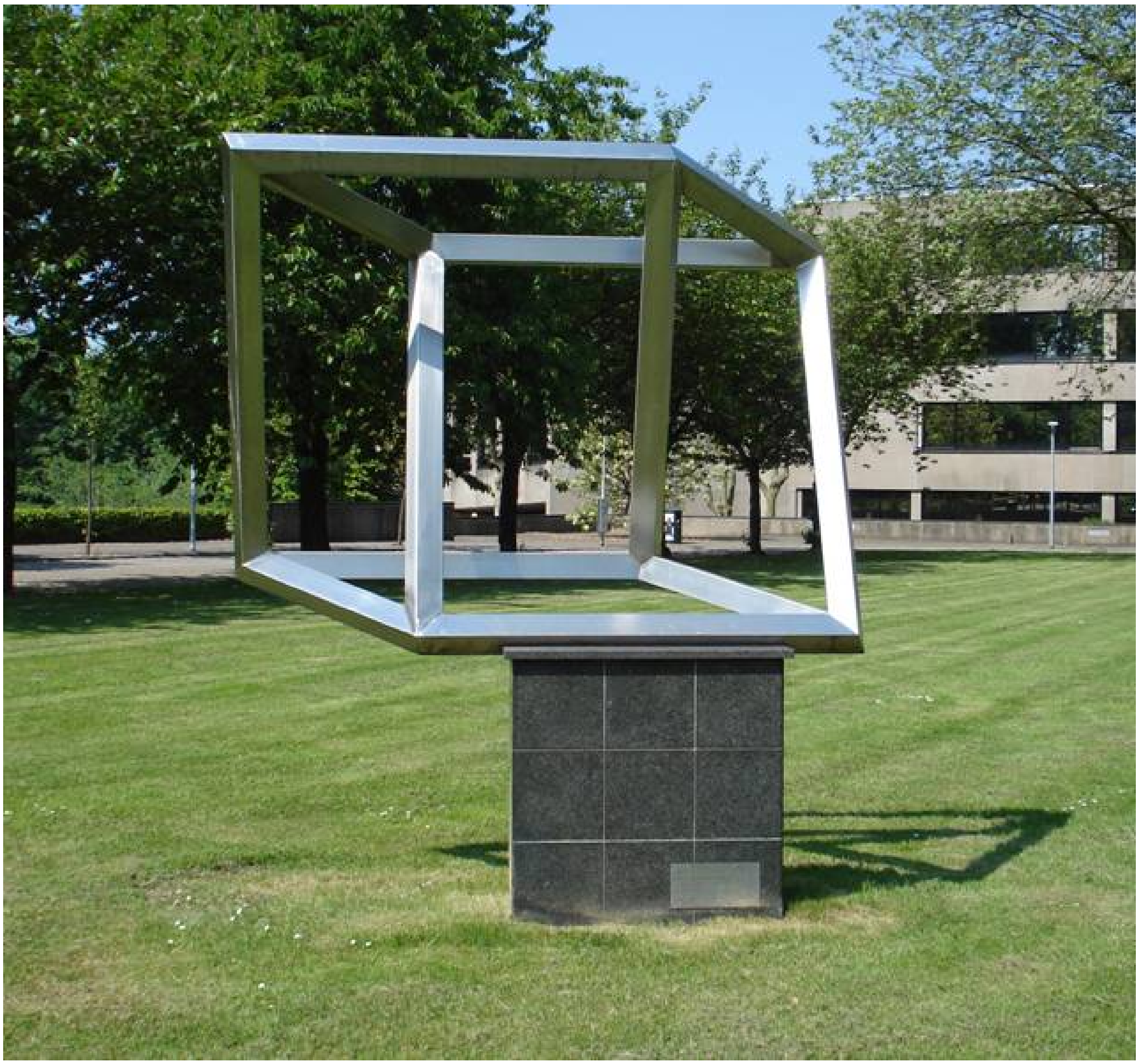

Figure 10: *The Impossible Cuboid* (1988, stainless steel, location: Erasumus University Rotterdam [3])

4 Miter Joints with More Than Three Branches

When four beams of the same cross section meet in a point, the situation becomes even more complicated. In this case, there are 6 pairs of binary miter joints to consider. Alternatively, one can first join three beams by a ternary miter joint and then try to fit in the fourth beam. This fourth beam needs to form matching miter joints with each of the three other beams. In general, this cannot be made to work by rotating the cross section appropriately (as was possible for the ternary miter joints). The cross section and the joint angles need to agree 'intimately'.

In [4], the fish-like state graph of Figure 11 (top left) appears. It describes the possible behaviors of a delay-insensitive system with two input ports `a` and `b` and two output ports `d` and `e`. The system starts in the center state labeled 0, and each arrow corresponds to an event on the corresponding port. The two states labeled 1 are actually a single state.

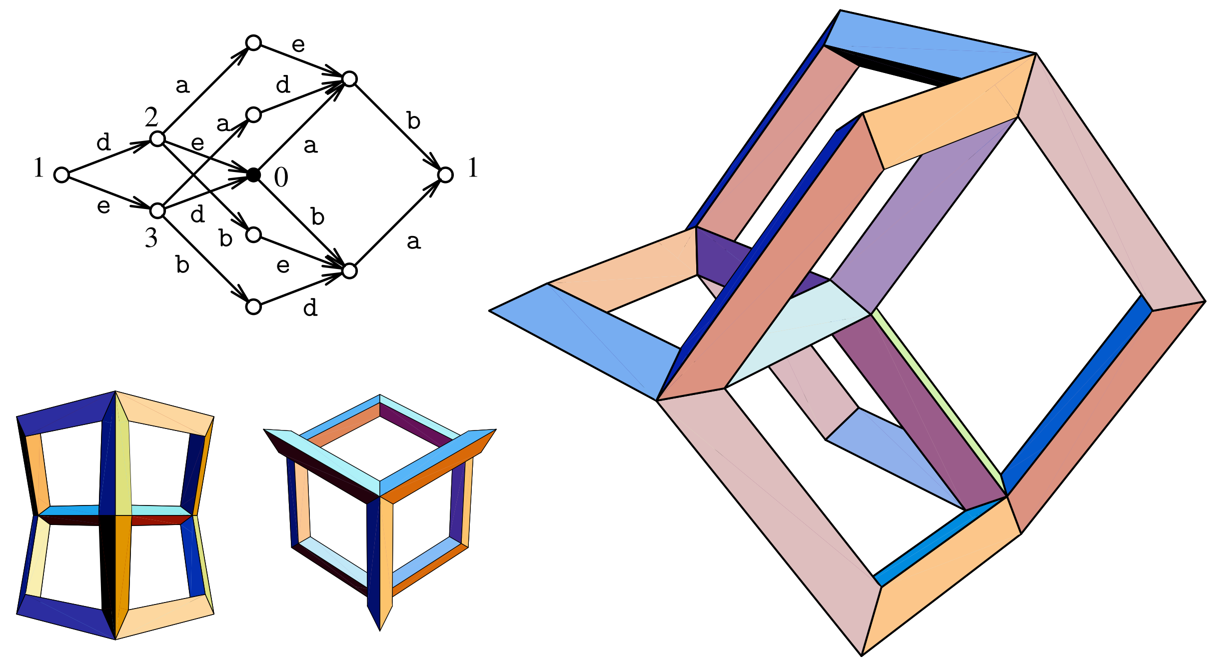

Figure 11: 'Fish' state graph (top left) and sculpture (three views) involving regular quaternary miter joints

The second author rendered this graph in 3D to avoid arrow crossings. The design is shown in Figure 11 and Figure 12 presents two wooden sculptures. Each of these sculptures consists of sixteen beams, having an equi-triangular cross section. They enjoy the 24 symmetries of the group S_4, which is also the symmetry group of the tetrahedron. There are six places where two beams meet and five places where four beams meet in regular quaternary miter joints. The design would not work with a square beam.

5 Conclusion

We have characterized the conditions to construct branching miter joints that connect multiple beams having the same cross section, in such a way that their longitudinal beam edges match at the joint. In contrast to the binary miter joint (connecting two beams), the branching miter joint imposes restrictions on the combination of joint angles and longitudinal rotation of the cross section. These branching miter joints can be applied in artwork, as we have illustrated.

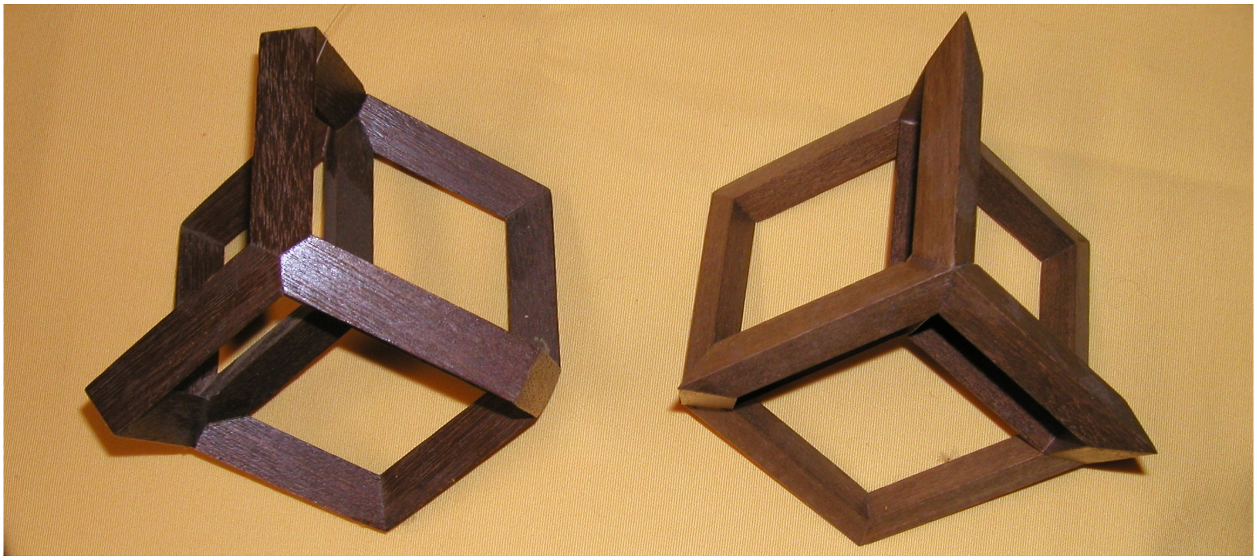

Figure 12: Fish sculptures (1994, wood, two versions, with beams rotated over 60°)

Note that the fractal trees designed by the second author do *not* involve ternary miter joints as treated in this paper. In those fractal trees, the cut faces do not contain the point where the center lines of the beams meet. Rather, the thicker branch is treated as a pair of thinner parallel beams, each connecting to a thinner branch through a binary *skew* miter joint. This will be presented in a future paper.

Acknowledgments All artwork and pictures were made by the second author. The illustrations were made with Mathematica and Xfig.

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