The Vitruvian Figure of Eight

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Abstract

The remarkable hidden symmetry of the Bernoulli lemniscate appeals to the mind and eye alike, and presents an opportunity to straddle the line between art and mathematics.

1 Introduction

The best known plane curve resembling the symbol for infinity ∞ is the *lemniscate of Bernoulli*. It is named after James Bernoulli, who considered the integral for the curve's arclength in his early work on elasticity theory (1694). The same arclength integral led to discoveries by Count Fagnano (1718) and Euler (1751) on the addition theorem for elliptic integrals, the key which opened up the theory of elliptic integrals and functions. Following Gauss's theorem (1796) on constructible polygons, Abel's result (1827) on subdivision of the lemniscate gave the curve a place in the history of algebra and number theory. (See [3], [5] and [6].)

It is surprising that a curve with such a history is not better known as a beautiful geometric object in its own right. The obvious elegance, symmetry, and association with infinity bestow on the lemniscate an undeniable mystique. In fact, hidden within itself, the curve carries a much richer structure. As explained in the last section, it is hardly a stretch to say that the lemniscate is intrinsically a *disdyakis dodecahedron*—dual to the great rhombicuboctahedron—with 48 triangular faces, 72 edges, and 26 vertices, which are permuted by the full octahedral group of symmetries. After providing brief mathematical and historical background, we offer a visual explanation of the lemniscate's structure, in Figures 3, 4. (Mathematical details are given in [2].)

2 Elementary Constructions: Linkages and the Like

The lemniscate equation may be written $(x^2+y^2)^2 = A(x^2-y^2)$ (or, in polar coordinates, $r^2 = A\cos 2\theta$). The pair of tangent lines to the double point at the origin is represented by the quadratic term $x^2-y^2=0$. With $A=2c^2$, the lemniscate has two additional x-intercepts $(\pm\sqrt{2}c, 0)$, and pair of *foci* $f_\pm = (\pm c, 0)$.

A simple "draftsman's tool" for drawing the lemniscate may be designed as in Figure 2 a). The device is in fact a special case of the *three rod linkage* considered by James Watt (1784), who was interested in converting rotational motion into linear motion. For the lemniscate, the two end rods have length $\sqrt{2}c$ and pivot about the foci $f_\pm$, while the middle rod of length $2c$ has no fixed end but is hinged to the first two rods. The pencil is mounted at the midpoint of the middle rod. As the end rods rotate (*oppositely*) in circles with centers $f_\pm$, the pencil traces out the lemniscate.

The ingenious linkage of Peaucellier (1864) achieved Watt's goal of interchanging rotational and linear motion. It would also enable a draftsman to construct (an arc of) the lemniscate from the rectangular hyperbola $x^2-y^2=c^2/2$ (and *vice versa*), as shown in Figure 2 b). The two rods in the figure with pivot at

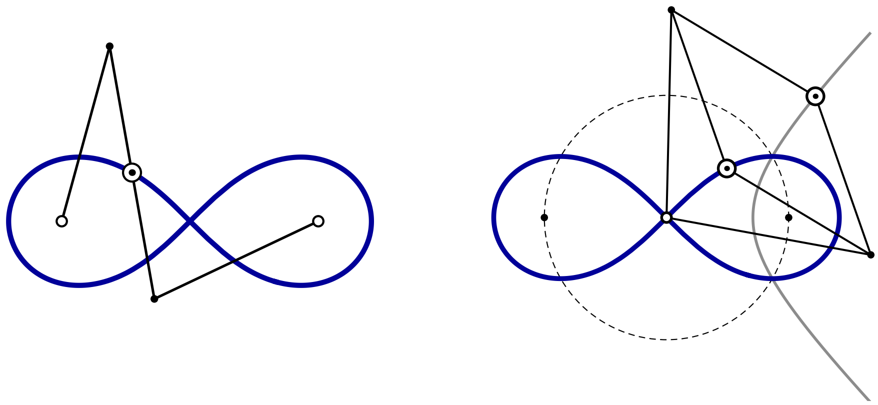

Figure 1 : *a) The infinity machine; b) Peaucellier's inverter.*

the origin have equal length $L > c$ and the remaining four rods have length $l = \sqrt{L^2 - c^2}$. The two "stylus" (circled) joints of the Peaucellier linkage maintain "inverse" positions with respect to the *circle of inversion* $x^2 + y^2 = c^2$ (dashed). As a transformation, inversion maps a point with polar coordinates (r, θ) $(r > 0)$ to the point $(c^2/r, \theta)$ on the same ray from the origin with (scaled) *reciprocal radius*. (For Watt and Peaucellier linkages, see [4], [7].)

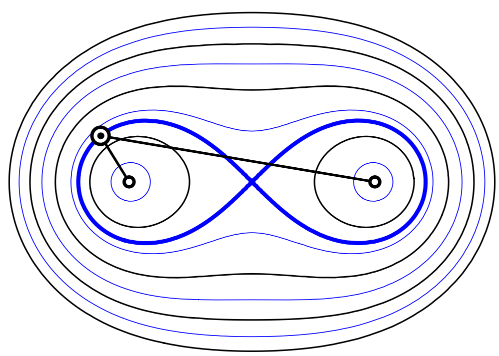

Figure 2 : *The lemniscate as the special Cassinian* $d_1 d_2 = c^2$.

The lemniscate and hyperbola related by inversion share the same pair of foci, $f_\pm = (0, \pm c)$. As for the meaning of *foci* we recall the *string construction of the ellipse*: Generalizing the circle (whose "two foci" coincide), the ellipse may be described as the locus of points the sum of whose distances from the foci is a given constant $d_1 + d_2 = 2C > 2c$. Likewise, the distances to the foci along the hyperbola have a constant *difference*, $d_1 - d_2 = 2C < 2c$. Finally, a *Cassinian oval* may be defined as the locus of points the *product* of whose distances from the foci $f_\pm = (0, \pm c)$ is a constant $d_1 d_2 = C^2$. The lemniscate is the Cassinian with $C = c$; smaller values $0 < C < c$ give pairs of "orbits", one around each sun $f_\pm$, while larger values $C > c$ give single orbits around the double star $f_\pm$. For sufficiently large $C > c$, the Cassinian is indeed oval, and was the astronomer Cassini's idea (1680) for planetary orbits in a (single star) solar system.

3 Alberti's Veil and the Extended Complex Plane

For nearly two centuries mathematicians have known that an algebraic curve is best understood in the complex projective setting, where one may take full account of all the complex and infinite points on the curve, not just the "visible" (real, finite) ones. It is hard to imagine how mathematicians could have reached this insight without the Renaissance development of perspective and all the geometric ideas which flowed from it. (See [1] and [6] for different aspects of this very substantial connection between art and mathematics.)

We will not explain in general how such a curve may be regarded as 2-dimensional (Riemann) surface in a 4-dimensional space, and how it may inherit also geometric structure from the latter space (the *complex projective plane*). But we will indicate, more concretely, how the full lemniscate looks topologically like a sphere, and has *rotational symmetries* familiar to us from our experience in the 3-dimensional, physical world. To do so, we first need to consider some key complex functions which describe the lemniscate.

To begin, we reinterpret the relationship between lemniscate and hyperbola. From now on, for notational simplicity, we take $c = 1$. Introducing the complex variable $z = x + iy$, the lemniscate and hyperbola are interchanged by *complex inversion* $z \mapsto \mathscr{I}(z) = 1/z$. We note that $\mathscr{I}(z)$ differs from circle inversion by reflection in the x-axis, an obvious symmetry of both curves (given now by *complex conjugation* $z \mapsto \overline{z} = x - iy$). With special definitions $\mathscr{I}(0) = \infty$, $\mathscr{I}(\infty) = 0$, $\mathscr{I}(z)$ takes the unit circle to itself and interchanges "inside" and "outside". Although $\mathscr{I}(z)$ distorts Euclidean distance in the plane, it preserves angles, and is as nice a transformation as one could ask for; in fact, $\mathscr{I}(z)$ may be understood as a rotation of the sphere!

What is required here is exactly *Alberti's veil* of Renaissance perspective. In mathematics, the standard correspondence of points of the sphere to points in the plane is known as *stereographic projection* from the north pole (the *eye*). Here we require the "reverse" application of the method of Alberti's veil to transfer features in the complex plane $\mathbb{C}$ (the *veil*) to features on the unit sphere $X^2 + Y^2 + Z^2 = 1$ (the *scene*, which in this case lies both in front of and behind the veil).

More explicitly, stereographic projection $\rho : S^2 \to \mathbb{C}$ is defined by considering downward sloping rays from the north pole $(0,0,1)$; the ray will intersect the sphere S^2 at a second point P and this point is mapped to (or identified with) the point of intersection of the ray with the equatorial plane $Z = 0$ (which is identified with $\mathbb{C}$). The north pole itself is sent to ∞ by a horizontal ray. From this "spherical perspective", complex inversion $\mathscr{I}(z)$ simply rotates the sphere by 180° about the X-axis.

Building on $\mathscr{I}(z)$, we will also make essential use of the *Joukowski map* $j(z) = \frac{1}{2}(z + \frac{1}{z})$, named after Zhukovsky (1847–1921) for his studies of airflow around obstacles. The key property of $j(z)$ is that it defines a smooth, angle-preserving mapping of the exterior (or interior) of the unit disc onto the *slit domain* obtained by removing the interval $[-1,1]$ from the complex plane. In general, such *conformal maps* transform 2-dimensional ideal fluid flow in one region into ideal flow in another region. In the case of $j(z)$, uniform, linear flow to the right is transformed into flow around an obstacle with circular cross section $|z|^2 = x^2 + y^2 \leq 1$. An additional application of $j(z)$ then transforms the latter into a flow around an airfoil (wing cross section). A nice variety of airfoils may be obtained as j-images of circles—this is the beauty of Zhukovsky's construction. (See [4] for discussion of $\mathscr{I}, \rho$, and flows around obstacles, emphasizing geometry.)

4 Metamorphosis of the Disdyakis Dodecahedron

In this section we rotate our lemniscate 90°—by change of sign, $A = -2$—so that it stands upright like a figure of eight 8. Correspondingly, we use the (conjugated) Joukowski map $j_{-}(z) = \frac{1}{2}(z - \frac{1}{z}) = -i\,j(iz)$, below. We give a (*clockwise*) "storyboard" explanation of the symmetry and structure of

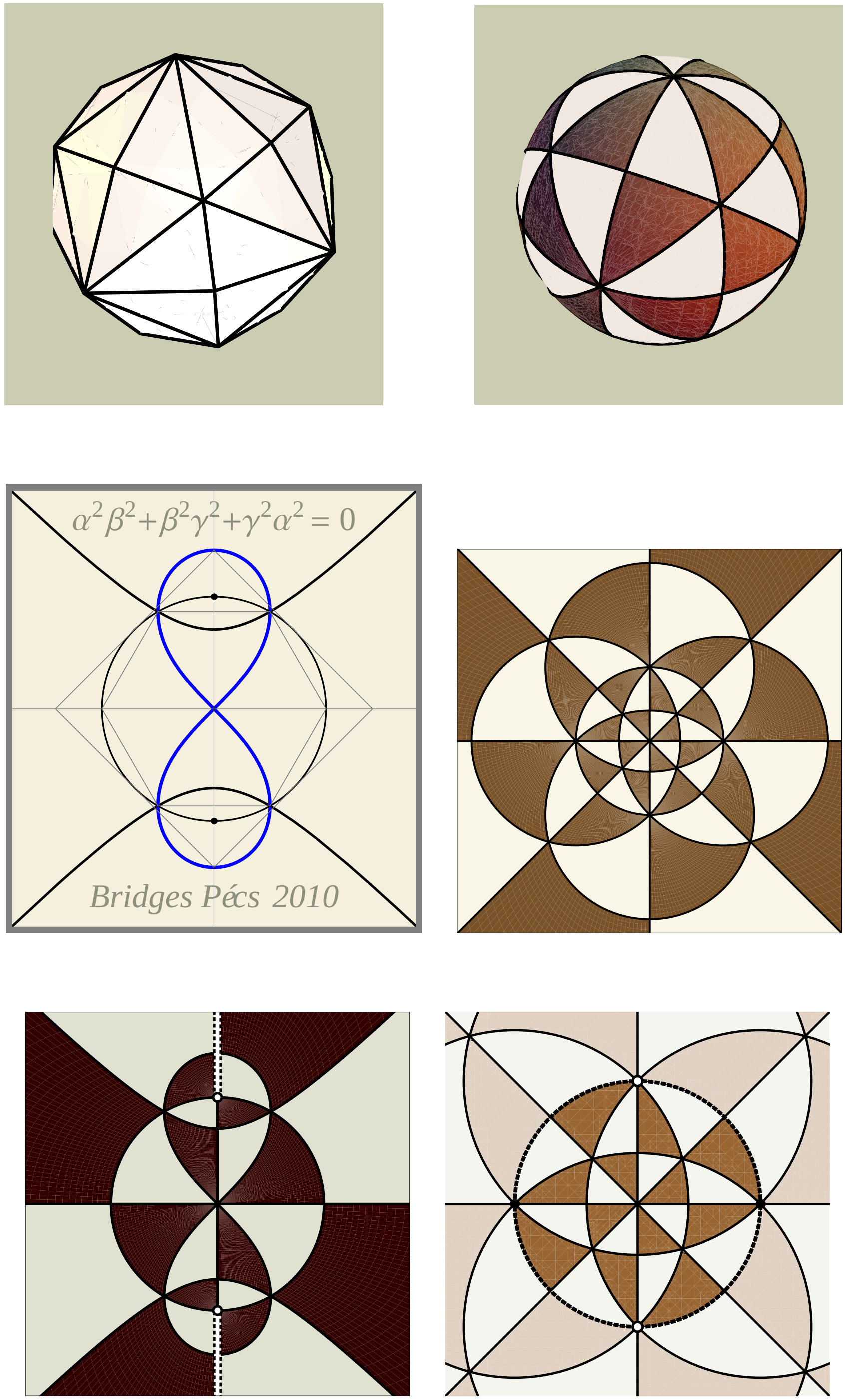

Figure 3 : *Metamorphosis of the Disdyakis Dodecahedron (clockwise from upper left).*

the lemniscate as a disdyakis dodecahedron, based on the mappings discussed above, and radial projection onto the sphere. The following comments may help to explain the *metamorphosis of the disdyakis dodecahedron*, Figure 3, in more mathematical terms:

1. Radial projection of the disdyakis dodecahedron (upper left) gives the tiled sphere (upper right), with 48 congruent spherical triangles, with *geodesic edges* and angles $45^\circ,\ 60^\circ,\ 90^\circ$.
2. Stereographic projection from the north pole gives the triangulated complex plane (middle right); the angles are the same and the non-straight edges of the "triangles" are arcs of circles.
3. The (extended) complex plane consists of the triangulated unit disc D (lower right) together with its *congruent image* under complex inversion $\mathscr{I}(z)$, the exterior E of the unit circle.
4. The inverted Joukowski map $\mathscr{J}(z) = \mathscr{I}(j_-(z)) = \frac{2z}{z^2-1}$ takes D onto the "slit plane" S_1 (lower left) and E onto an identical copy $S_2 \simeq S_1$. The slit (dashed) extends from $z = i$ up to ∞, and from $z = -i$ down to ∞, and may be thought of as a "collapsed circle", folded at $\pm i$, with left and right edges both containing ∞, the images of the points ± 1 in the previous figure. Thus, the $\mathscr{J}$-image of the entire complex plane consists of two "sheets" (only one of which is shown), which are zipped together along the slit.
5. The *Vitruvian figure of eight* (middle left) consists of lemniscate, circle, hyperbola, and x, y axes. These were regarded above as separate plane curves (related by Peaucellier); now they are but traces of the full lemniscate (with equation $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = 0$, in suitable complex coordinates), each trace a "mirror" of reflection symmetry. But the enigmatic Vitruvian eight hides the existence of *the second sheet* (glued to the back?), without which the full symmetry is lost (leaving square and hexagon still visible as cryptic scaffolding).

Not to get too serious, we have invoked *Vitruvian Man* (1487), Leonardo Da Vinci's iconic study of proportion, symmetry and hidden mathematical meaning in the figure of man, inspired by the writings of the Roman architect Vitruvius. But mathematical curiosity played no small part in the Renaissance imagination; Da Vinci (a brilliant student of polyhedra) would surely have allowed even the figure of eight to hold meaning, beauty, and a hint of mystery.

5 The Lemniscate for its Own Sake

Works of Gauss (1827) and Riemann (1854) on intrinsic differential geometry made it possible to discuss the *shape* of a curved surface without the need to view the surface "from the outside". Given a *Riemannian metric*, one can measure distances and angles on the surface, define *geodesics* (paths of shortest length), compute the surface's *Gaussian curvature* K, etc. The same idea, suitably generalized, was exactly what Einstein required for his interpretation of gravity in terms of the intrinsic curvature of 4-dimensional space-time, and the unified treatment of light rays and inertial motion of particles as geodesics.

What is the lemniscate's intrinsic geometry as a surface (complex curve)? In its own right, the lemniscate may be viewed as a topological sphere (free of self intersections), with Riemannian metric inherited from complex projective space. At some points the lemniscate is positively curved ($K > 0$) like a sphere of radius $R = 1/\sqrt{K}$; at others, the lemniscate is negatively curved ($K < 0$), like a potato chip.

The two contour plots of K in Figure 4 reveal the remarkable symmetry of the lemniscate. Here, the standard sphere serves as reference space for plotting regions $K_j < K < K_{j+1}$ (left), and the same regions are shown stereographically projected onto the plane (right); in both plots, K has constant value along the (dark) "boundary curves". We locate the curvature function's critical points with respect to the *cuboctahedron* (!) on the left: Six maxima $K_{max} = 2$ at centers of fourfold rotational symmetry; eight minima $K_{min} = -7$ at centers of threefold rotational symmetry; twelve saddle points $K_{saddle} = -1/4$ at the centers of twofold rotational symmetry. (The latter K-values are unexpectedly simple.)

The $26 = 6 + 8 + 12$ critical points are the vertices of a *Riemannian disdyakis dodecahedron*. For the lemniscate also turns out to have exactly nine simple closed geodesics of reflection symmetry (represented by circles and lines on the right side of Figure 4), which are subdivided by the 26 vertices into the 72 edges of 48 congruent geodesic triangles; the latter $45°, 60°, 90°$ "tiles" are simply transitively permuted by the full octahedral symmetry group of the lemniscate, and define a non-constant curvature analogue of the $*432$ tiling of the sphere already pictured in Figure 3.

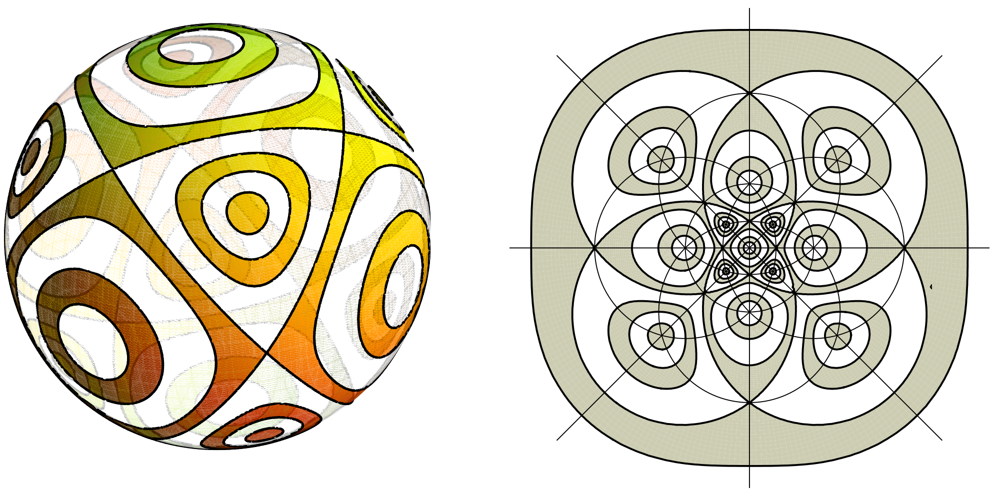

Figure 4 : *Gauss curvature of lemniscate on reference sphere and plane.*

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