The Sunflower Spiral and the Fibonacci Metric

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Abstract

The sunflower spiral is a well known and attractive pattern often seen in the arrangement of leaves or other features in plants (phyllotaxis), with close connections to the Fibonacci sequence. We describe an interesting colouring of the points of the sunflower spiral, involving a "metric" on the positive integers which counts the number of distinct non-consecutive Fibonacci numbers needed to sum to a given number. We also explain some initially surprising features of the resulting coloured pattern.

1 The Sunflower Spiral

Definition. The **sunflower spiral**[1] is given by the function $S : \mathbb{N} \to \mathbb{R}^2$, written in polar coordinates as $S(n) = (r(n), \theta(n)) = (\sqrt{n}, 2\pi\varphi n)$, where $\varphi = \frac{\sqrt{5}-1}{2} = \Phi - 1$, and $\Phi = \frac{\sqrt{5}+1}{2}$ is the golden ratio.

See figure 1. The choice of function $r(n)$ (so that the points lie on a Fermat spiral) gives an equal area packing[2]. However, the observations related to the colouring described in this paper will also hold for other choices of $r(n)$.

2 The Fibonacci Metric

The following function $M : \mathbb{N} \to \mathbb{N}$ was communicated to the author by Ravi Vakil, and is a feature of the cover image (designed by the author of this paper) of the second edition of his book [4]. Zeckendorf's theorem (see for example [1]) states that every positive integer can be represented uniquely as a sum of distinct Fibonacci numbers, given the additional condition that no two consecutive Fibonacci numbers are used.

Definition. We define the **Fibonacci metric**, $M(n)$, to be the number of terms in the Zeckendorf representation of n.

Zeckendorf's theorem implies that the greedy algorithm to calculate $M(n)$ works, taking the largest Fibonacci number less than the remaining number at each stage. We colour nodes of the sunflower spiral according to the value of the function, as described in the caption for figure 2. See figure 3 for the continuation of the pattern. There are two interesting features that we see in figure 3 which deserve some explanation:

[1]This is our name for the spiral. There doesn't seem to be a generally accepted name in the literature. Dixon [2] uses the term "true daisy". Vogel [5] refers to the "Fibonacci angle" between nodes but does not name the spirals with that angle, which is the key feature for our purposes.

[2]This means that the ratio of the number of points in a circle centered at the origin to the area of that circle approaches a limit as the radius goes to ∞, see Ridley [3]. Ridley also shows that among patterns with $r(n) = \sqrt{n}$ and $\theta(n) = \lambda n$ for some $\lambda \in \mathbb{R}$, the sunflower spiral gives the most efficient packing pattern, meaning that the infimum of distances between nodes is maximised.

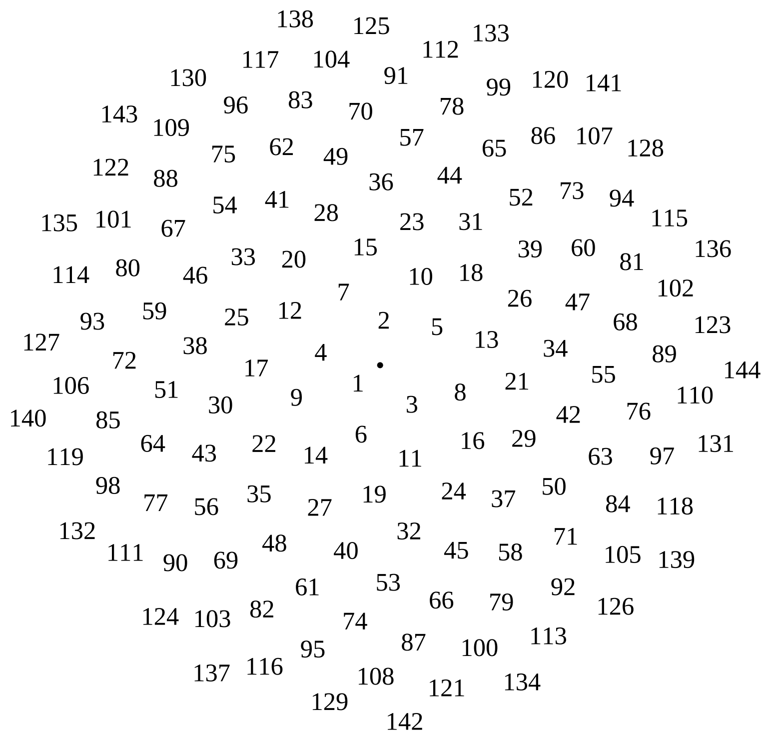

Figure 1 : *The positions of the first 144 points of the sunflower spiral. The origin is marked with a dot.*

(A) There are bright radial spokes, in particular the "Fibonacci spoke" at $\theta = 0$, near which the Fibonacci numbers themselves lie.

(B) There are bright circular "tree rings", most obviously one around the radius of each Fibonacci number, although there are weaker tree rings at other radii.

We explain (A) as follows. The Fibonacci numbers lie near the ray $\theta = 0$ because of the following argument: A calculation using the identity $F_k = \frac{\Phi^k - (1-\Phi)^k}{\sqrt{5}}$ shows that $F_k - \varphi F_{k+1} = (-\varphi)^{k+1}$. Then $\theta(F_{k+1}) = 2\pi\varphi F_{k+1} = 2\pi\left(F_k - (-\varphi)^{k+1}\right)$. Since $|\varphi| < 1$, $(-\varphi)^{k+1}$ is near zero for large k, and so $\theta(F_{k+1})$ is approximately a multiple of 2π, with the approximation improving exponentially with k.

Also making up the Fibonacci spoke are many relatively bright points at radii in between the radii of the Fibonacci numbers. These brighter points are sums of a small number of larger Fibonacci numbers: Since the larger Fibonacci numbers have angles closely approximating $\theta = 0$, sums of them will have angles which are the sum of those angles, and so will also be close to $\theta = 0$.

The other spokes result from rotations of the Fibonacci spoke: if we add a number n to all numbers corresponding to points near to the ray $\theta = 0$, then we get numbers corresponding to points which will be arranged near to the ray $\theta = 2\pi\varphi n$. The points corresponding to these numbers will be at worst $M(n)$ steps darker than the original points.

To explain (B), it is enough to note that the Fibonacci numbers are relatively dense early in the integers and become sparser later on, which implies that $M(n)$ will generally be small for small n. Just after a number m with small $M(m)$ (e.g. m a Fibonacci number) appears, there will be many numbers for which the minimal $M(n)$ is achieved using m and some small number of additional Fibonacci numbers, because $n - m$ is small.

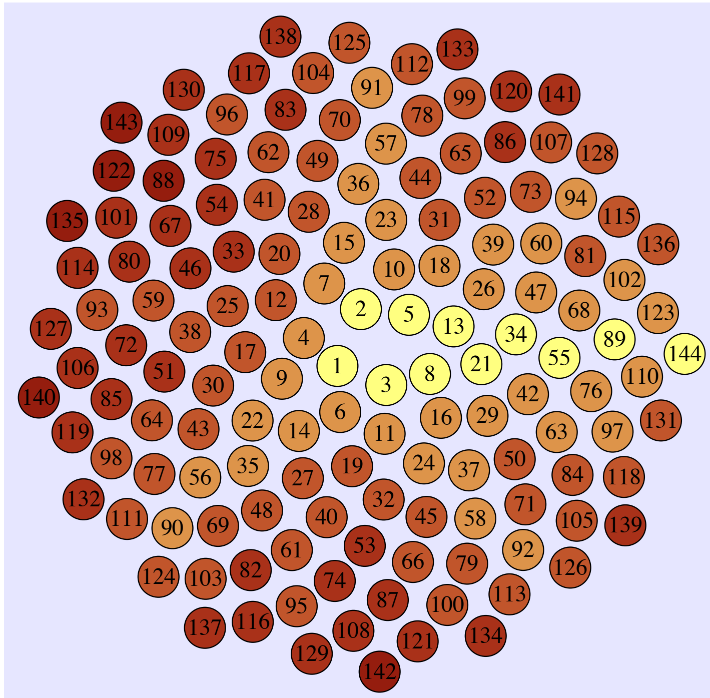

Figure 2 : *Nodes of the sunflower spiral up to* $n = 144 = F_{12}$*, coloured according to the Fibonacci metric. The brightness of the nodes is a decreasing function of* $M(n)$*. The brightest nodes are the Fibonacci numbers themselves, because* $M(F_k) = 1$.

These points appear at radii only slightly larger than that of m and so form the tree ring. An attractive feature of this pattern is the way that it highlights the positions of the Fibonacci numbers with the tree rings, so demonstrates visually that the Fibonacci numbers grow exponentially.

3 Conclusions and Further Research

It is well known that the pinecones and other objects following the sunflower spiral pattern exhibit spiral arcs in two directions, with the numbers of spirals being consecutive Fibonacci numbers. Colouring according to the Fibonacci metric reveals other subtle connections between the spiral and the Fibonacci sequence.

If we emphasise the set of nodes with a particular value of $M(n)$, for example as in figure 4, we see an interesting irregular but non random, perhaps fractal pattern. Can we describe and explain its features?

References

[1] J. L. Brown, Jr., *Zeckendorf's theorem and some applications*, Fibonacci Quarterly **2** (1964), 163–168.

[2] Robert Dixon, *Mathographics*, Dover Publications, 1991.

[3] J. N. Ridley, *Packing efficiency in sunflower heads*, Mathematical Biosciences **58** (1982), no. 1, 129–139.

[4] Ravi Vakil, *A Mathematical Mosaic: Patterns & Problem Solving*, 2nd ed., Brendan Kelly Publishing, 2007.

[5] Helmut Vogel, *A better way to construct the sunflower head*, Mathematical Biosciences **44** (1979), no. 3-4, 179–182.

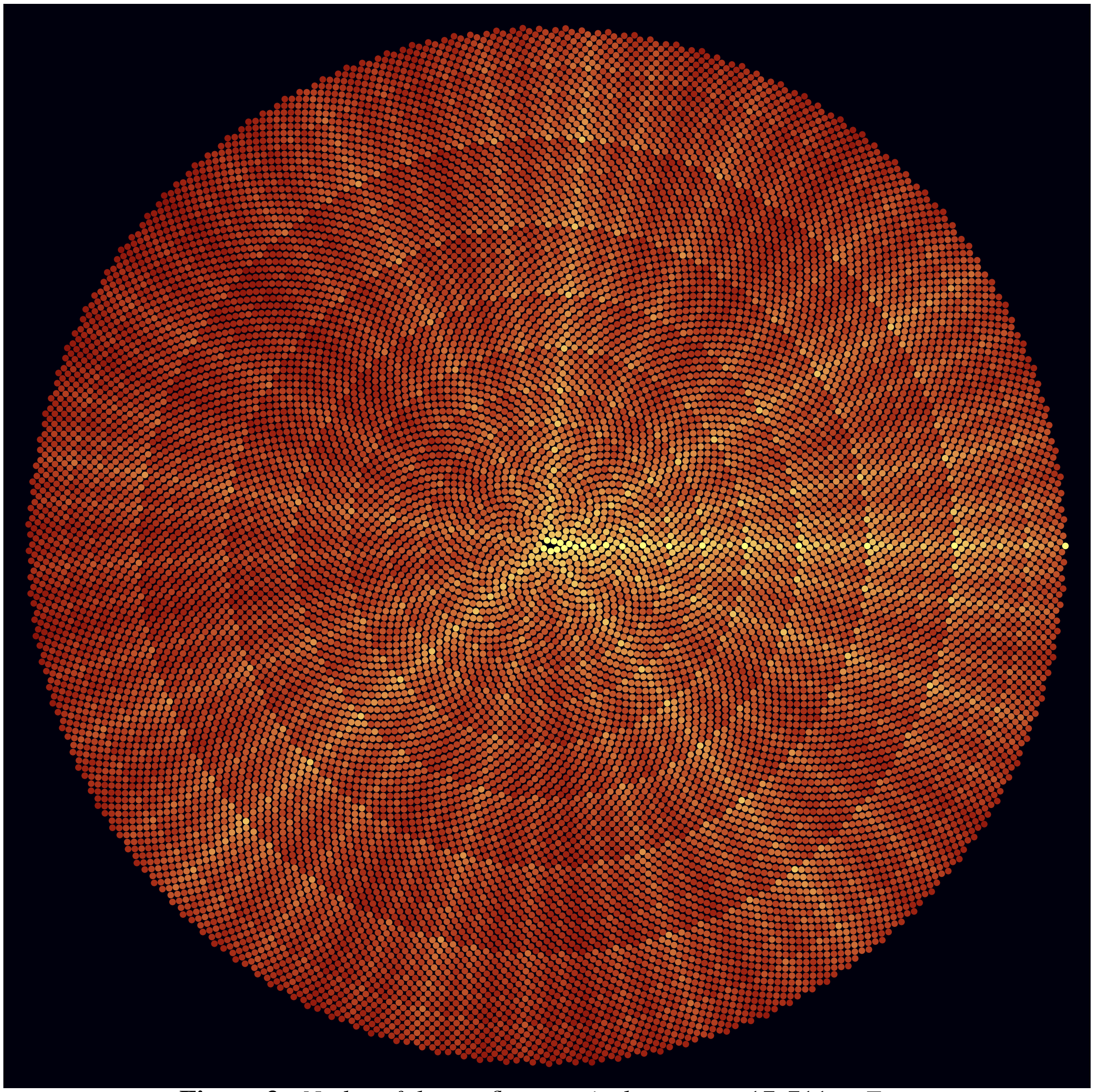

Figure 3 : *Nodes of the sunflower spiral up to $n = 17,711 = F_{22}$.*

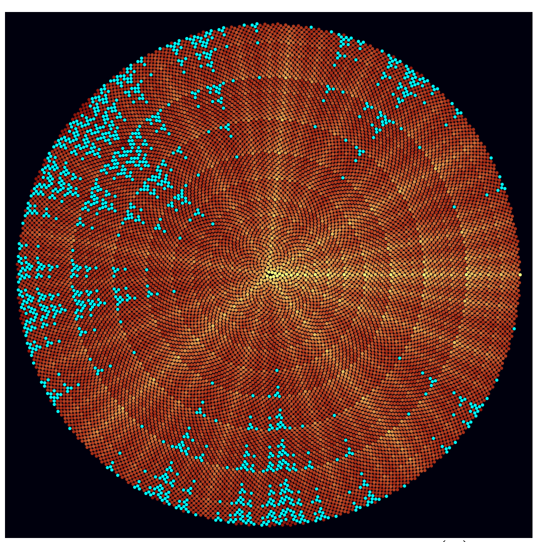

Figure 4 : *The image from figure 3, nodes with $M(n) = 8$ highlighted.*