Ad Quadratum Ad Infinitum

Eliana Manuel Pinho*
Centro de Matemática
and Faculdade de Arquitectura
Universidade do Porto
empinho@gmail.com

Abstract

We consider planar patterns with two colours, white and black, that are periodic along two non-colinear directions and such that the white and black regions are related by one Euclidean isometry. The fundamental cells of these patterns have half the area in white and the other half in black. In Roman mosaics with two colours exhibiting square-based repetitive patterns, the balance between the two colours is often solved via the *ad quadratum* geometrical construction. Several patterns observed in Roman mosaics with two colours, mainly from *Conimbriga*, are presented to illustrate this idea.

1 Two-Colour Periodic Patterns

In this section we present the definitions of periodic patterns and, in particular, the ones having two analogous one-colour regions. We can approach this problem in several ways. The approach we follow has the advantage of being easily adapted for patterns with k colours [2]. Moreover it can be used for patterns in n-dimensional spaces that are periodic along n non-colinear directions [6]. See [3] for the enumeration of the two-colour symmetry groups.

Periodic Patterns. Figure 1 summarises the main ideas of this paragraph for the readers not wishing to follow a formal mathematical approach. We consider planar patterns that are periodic along two non-colinear directions. The set of all the periods of these patterns forms a lattice $\mathscr{L}$, a subset of the Euclidean plane $\mathbf{R}^2$, generated over the integers by two linearly independent vectors $l_1, l_2 \in \mathbf{R}^2$, which we write $\mathscr{L} = \{l_1, l_2\}_{\mathbf{Z}} = \{m_1 l_1 + m_2 l_2 : m_1, m_2 \in \mathbf{Z}\}$. The vectors l_1, l_2 that generate $\mathscr{L}$, define a parallelogram called the *fundamental cell* of the lattice: $D = \{t_1 l_1 + t_2 l_2 : t_1, t_2 \in [0,1)\}$. The fundamental cell is the region whose *ad infinitum* repetition fills the plane with the pattern.

Beyond periodicity, these patterns can exhibit more symmetries, *i.e.*, they can remain invariant under the action of rotations, reflections or glide reflections. These transformations, including the translations, and their composition are *Euclidean isometries*, since they preserve the distance between any two points. The set of all the symmetries of a periodic pattern, define a *wallpaper group*, see [1] for a complete step by step study of the subject.

Two-Colour Patterns. Two-colour patterns are periodic patterns with two colours, say black and white, where the white and black regions are the same, up to isometries. See Figure 2 for an illustration. Formally, a planar pattern with two different colours, indexed by the set $\{1,2\}$, can be identified with a function $\xi : \mathbf{R}^2 \longrightarrow \{1,2\}$ where $\xi(x)$ is the colour of the point $x \in \mathbf{R}^2$. The region of the plane with colour i is the level set of ξ corresponding to the value i. If the colours are white and black, the white and black regions are the level sets $W = \{x \in \mathbf{R}^2 : \xi(x) = 1\}$ and $B = \{x \in \mathbf{R}^2 : \xi(x) = 2\}$.

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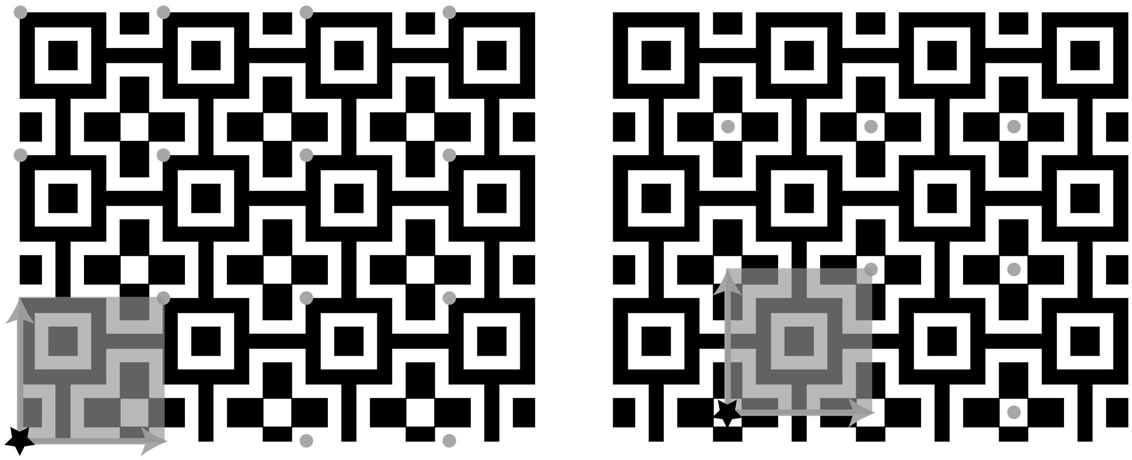

Figure 1: *We consider that the periodic patterns cover the whole plane and, thus, this figure depicts two small regions of an infinite pattern. In each region, the grey arrows represent two periods which define the lattice of all the periods (grey dots). They also define the grey parallelogram corresponding to a fundamental cell, a piece whose periodic repetition fills the pattern. If we choose a different origin (black star) then the lattice of periods undergoes a translation. In the second region, the choice of the fundamental cell highlights the symmetries of the pattern.*

Let ξ be a periodic pattern with two colours. We say that ξ is a *two-colour pattern* if there is an Euclidean isometry γ such that, for all $x \in \mathbf{R}^2$, $\xi(\gamma \cdot x) = \sigma(\xi(x))$, where σ is the permutation of the colours: $\sigma(1) = 2$ and $\sigma(2) = 1$. Therefore, for a two-colour pattern, we have $\gamma \cdot W = \{\gamma \cdot x : x \in W\} = B$ and, analogously, $\gamma \cdot B = W$.

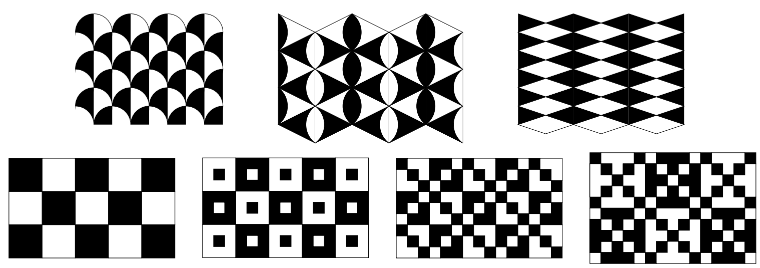

Figure 2: *Seven geometrical designs, existing in black and white Roman mosaics, that are two-colour patterns according to our definition: the white and the black regions are related by an isometry and, thus, fill the same area.*

Let D be the fundamental cell of a two-colour pattern. As a consequence of the definition, the areas of each colour in the fundamental cell, $D \cap W$ and $D \cap B$, are the same and, thus, the area of D doubles the area of the fundamental piece for each colour. This establishes a strong connection between the two-colour patterns and the geometrical problem of doubling the area of a given shape.

2 *Ad Quadratum*

Romans used widely the *ad quadratum* geometrical construction in decorative motifs and in architecture [4]. The *ad quadratum* construction solves geometrically the problem of finding a square that doubles the area of a given square, see Figure 3. Doubling the area of a figure is a inner feature of the two-colour patterns and, for square-based designs, the *ad quadratum* geometrical construction arises naturally.

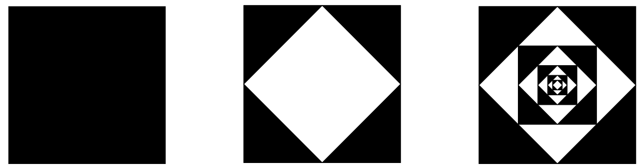

Figure 3: *The ad quadratum geometrical construction, in the middle of the figure, associates the big black square with the white square having half its area. The sides of the two squares are related by a* $\sqrt{2}$ *factor. On the right hand side, we present a succession based on the ad quadratum construction.*

We observed repetitive geometrical patterns in black and white Roman mosaics and found a huge diversity of motifs that relay in the *ad quadratum* geometry. Many of the patterns presented below can be seen in Roman mosaics in *Conimbriga*, Portugal (see photographs in [5]). In Figure 4 we present some repetitive patterns that exhibit an *ad quadratum* structure but are not two-colour patterns.

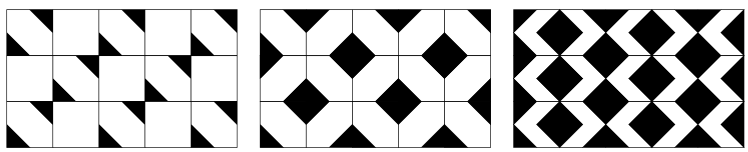

Figure 4: *Three geometrical designs, existing in black and white Roman mosaics, that are not two-colour patterns according to our definition (the white and the black regions occupy areas not related by an isometry) but are based in the ad quadratum geometrical construction.*

In Figures 5 and 6 are depicted several designs that are two-colour patterns and where the boundaries between colours are based on the *ad quadratum*. For each pattern, the reflections in the dashed lines are reflections that either leave the pattern invariant or swap colours. The shaded square is a fundamental cell. The particular features of each pattern are described below.

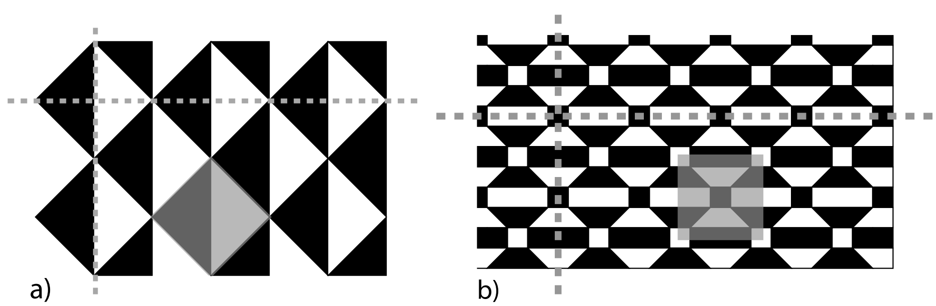

Figure 5: *Two-colour patterns from Roman mosaics that are based in the ad-quadratum geometry. The shaded regions are fundamental cells.*

Figure 5. a) To the vertical dashed line corresponds a reflection that swaps colours and, thus, ensures that the white and black regions have the same area. The horizontal line corresponds to a reflection that leaves the pattern invariant. The boundaries between the two colours define a square and one of its diagonals. b) The diagonals of the shaded square are, partially, the boundaries between the white and black regions. The dashed lines correspond to reflections that leave the pattern invariant. Consider a translation from a small white square to a small black square, for example, the vector corresponding to half the diagonal in the

fundamental cell. This translation swaps the colours of the overall pattern and, thus, ensures the similar roles of both the white and the black regions.

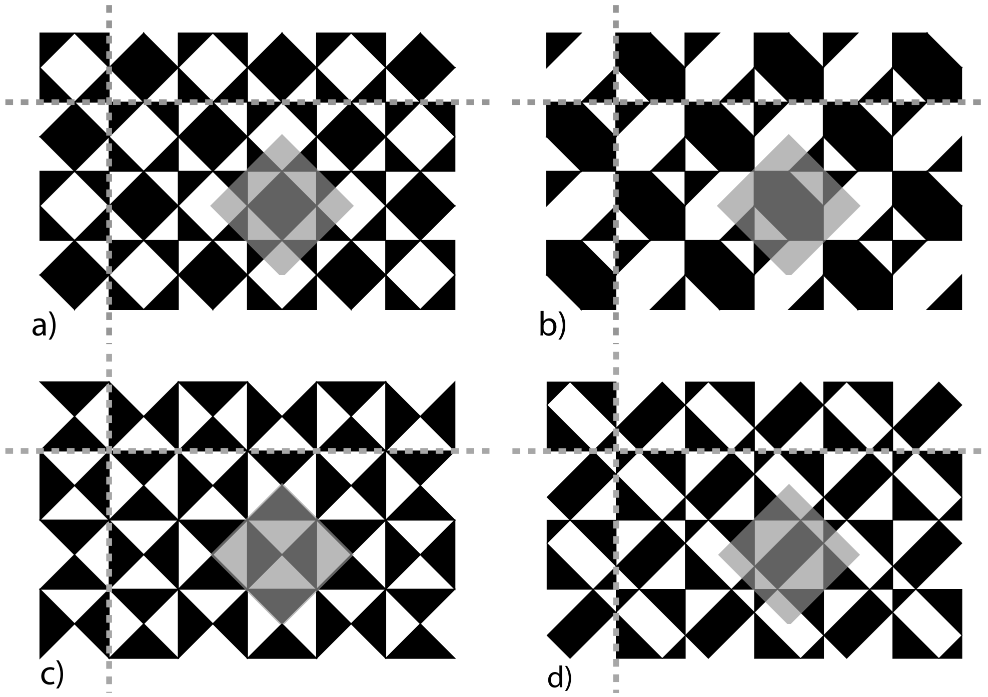

Figure 6 : *Four repetitive designs, observed in Roman mosaics, that are two-colour patterns.*

Figure 6. Both dashed lines are associated with reflections that swap colours, ensuring that the areas of the white and the black regions will be the same, on the whole pattern. a) The pattern is a periodic repetition of the ad quadratum geometrical construction. This can be perceived either in the square isolated by the dashed lines (top left) or in the fundamental cell (shaded square) where we see a sequence of a white square and a inner black square. b) Comparing this pattern with the previous one, especially their fundamental cells, helps clearing out the role of the incomplete ad quadratum construction in the definition of the boundaries between colours. c) and d) The ad quadratum geometry is also present in these patterns. Choosing different fundamental cells will enhance either the sides or the diagonals of the squares involved in the pattern.

References

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