Minimal Flowers

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Abstract

This paper describes my sculptures *Minimal Flower 3*, an homage to Brent Collins, and its new cousin, *Minimal Flower 4*. They are both constructed as minimal surfaces spanning certain knotted boundary curves, with three-fold and four-fold rotational symmetry, respectively.

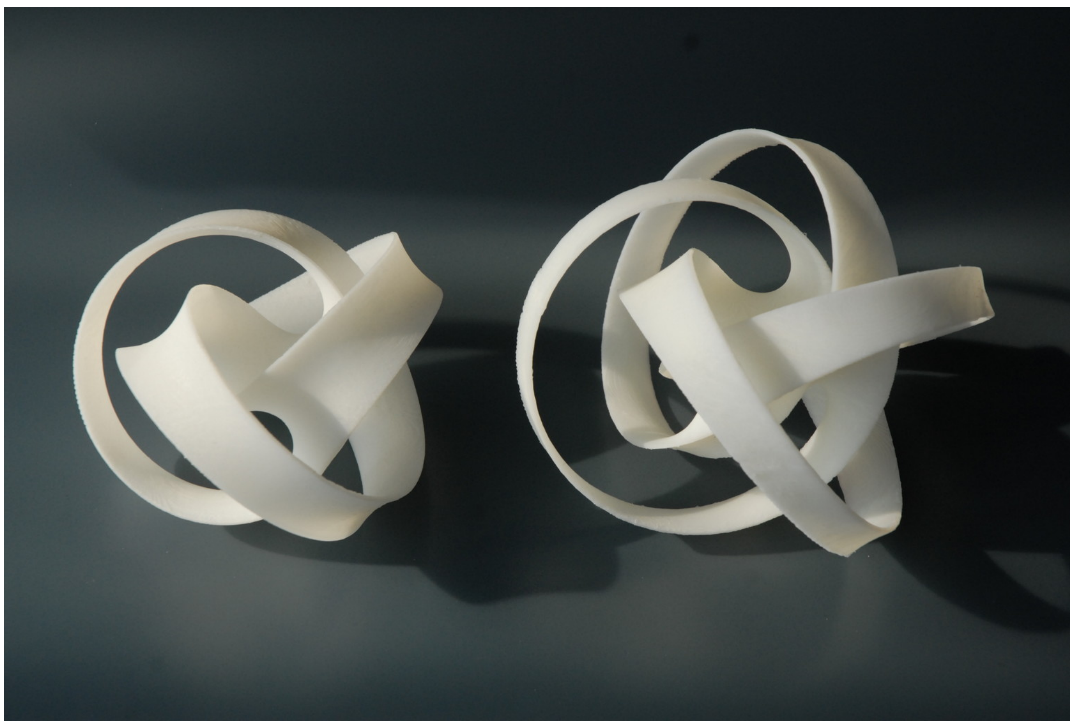

Figure 1: *Fused deposition models of* Minimal Flower 3 *(left) and* Minimal Flower 4 *(right). These models (at the 12 cm scale) were printed at the 3D Labor [6] in the mathematics department at TU Berlin.*

1 Brent Collins: *Atomic Flower II*

I first met the American sculptor Brent Collins at Bridges 1999. His sculptures often feature surfaces of negative Gauss curvature, including ribbons with a U-shaped cross-section (as in *Pax Mundi*) and bent towers of saddles or monkey saddles (as in many of his collaborations with Carlo Séquin). At Bridges 1999, he described his new two atomic flowers as the result of merging these two paradigms [2]: they each consist of three ribbons connected via a central monkey saddle. *Atomic Flower II* – not shown in his paper but shown at the conference in the wooden master copy (Figure 2, left) – implements this idea with perfect 3-fold rotational symmetry, particularly appealing to me.

The next summer I met Collins again at Bridges 2000, where he presented new work like *Music of the Spheres* [3]. On the way home I also got to see his large exhibit at the Albrecht–Kemper Museum in St. Joseph, Missouri, which included *Atomic Flower II*. That same year, I bought a bronze cast (Figure 2, right) of *Atomic Flower II*. (As with Collins' other bronzes, the cast is by Steve Reinmuth.)

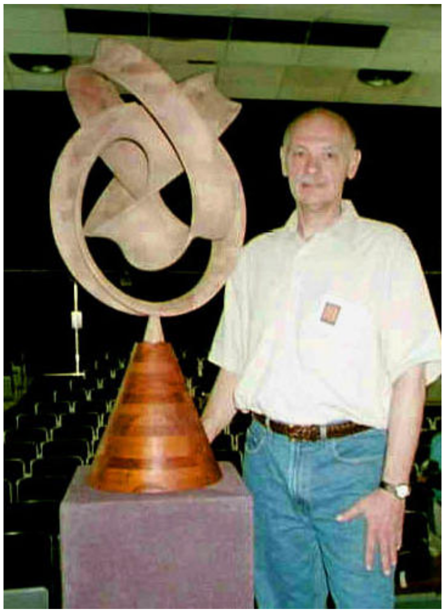

Figure 2: *Left: at Bridges 1999, Brent Collins presented the wooden master of* Atomic Flower II. *(Photo courtesy of Carlo Séquin.) Right: my bronze cast of* Atomic Flower II.

2 Sculpture via Geometric Optimization: *Minimal Flower 3*

Many of Collins' pieces have been described as being like minimal surfaces or soap films. Intrigued by this, I set about trying to make a computer model of a minimal surface with the topology of *Atomic Flower II*.

The first step was to design the boundary curve, the wire frame across which to span the virtual soap film. I started with three helices lying on mutually perpendicular cylinders, each extending almost two full turns; these are joined by short connecting arcs to make a single smooth curve. If we imagine the cylinders along the coordinate axes, then the entire configuration has the three-fold rotational symmetry that rotates x into y into z into x. Indeed, it also has two-fold rotational axes perpendicular to the three-fold one, meaning that its entire symmetry group is 322 in the Conway–Thurston notation. To improve the aesthetics, I modified the height function along each helix to be cubic instead of linear: this makes the ribbons wider near the central monkey saddle, and a bit narrower in the outer parts.

As with any knotted curve, there are presumably many different soap films that can span this particular boundary as minimal surfaces. Although the `Surface Evolver` [1], a computer program from Ken Brakke, will minimize area numerically to simulate a soap film or minimal surface, we have to find an initial spanning surface with the desired 322 symmetry and the correct (intrinsic and extrinsic) topology. The initial geometry on the other hand can be crude and will be refined automatically in the area-minimization process. Here, we can start with a planar hexagon in the middle (which becomes the monkey saddle) and attach three long orientation-reversing strips. Each strip has its ends glued to two adjacent sides of the hexagon, but the strip itself loops around the entire hexagon, leaving one side upwards and returning to the adjacent side from below. (See Figure 3, left.)

Letting the `Surface Evolver` minimize area, I got a surprise: the minimization process breaks the symmetry. The monkey saddle moves up along the three-fold axis, breaking the two-fold symmetry and resulting in a surface with just 33 symmetry (Figure 3, right). This is perhaps the soap film of least area spanning the given boundary, but its lower symmetry means it isn't the one I wanted.

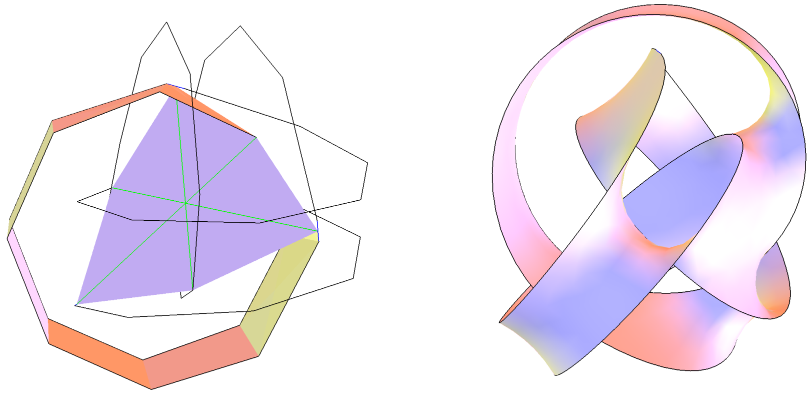

Figure 3: *Left: the initial surface consists of a flat central hexagon with three long strips, each joining two adjacent sides. Right: if we minimize area without enforcing symmetry, the monkey saddle moves along the axis of three-fold symmetry – here towards the right. The result has three-fold but no two-fold symmetry.*

To overcome this problem, I declared the three main diagonals of the initial flat hexagon to be fixed lines – they are the two-fold rotational axes that we want to preserve, and the desired minimal surface passes exactly through them. Minimization now resulted in a soap film with the desired 322 rotational symmetry (and about 8% more area).

Although I sent the resulting datafile to Carlo Séquin to make a 3D print on his FDM machine (see the photo in [4]), the surface was not yet aesthetically satisfying. A minimal surface always has nonpositive Gauss curvature, but the fact that the principal curvatures are equal and opposite means there is sometimes not as much negative Gauss curvature as one would like. In particular, in the surface I had designed, the long strips were too flat: one of their principal curvatures (along the strip) was essentially the curvature of the boundary helices, and so the other (across the strip) was equally small.

This presented me with a puzzle: how to improve the aesthetics while keeping the principle that the surface be determined by geometric optimization. My solution was to place the boundary curve symmetrically into Poincaré's conformal ball model of hyperbolic three-space. In this model (the two-dimensional version of which was used for M.C. Escher's famous *Circle Limit* prints) straight lines are represented by circular arcs perpendicular to the bounding sphere, and thus lines (especially near the boundary) are seen as bending inwards. Similarly, any minimal surface will appear to bend inwards more than a euclidean minimal surface would. With one free parameter (the size of the boundary curve within the model of hyperbolic space) for artistic control, I quickly found a surface I was happy with.

Of course, even if some sculptures (including *Atomic Flower II*) can be described as surfaces, a real sculpture is not an infinitesimally thin mathematical surface. It was straightforward to create two offset surfaces with a constant normal distance between them, but again this first mathematical solution was not aesthetically pleasing. The sculpture should be thinner near the boundary curve and get thicker as we move towards the center of the strips or towards the monkey saddle. To implement this idea, I again turned to geometric optimization. I took the doubled surface and blew a bit of (virtual) air between the two sheets. This results in a soap bubble bounded by two surfaces of constant mean curvature sharing the same boundary wire. (If we started with a flat disk of soap film on a circular wire boundary, we would end up with two spherical caps of soap film bounding a lens shape.)

Finally, I had an object I was happy with: this result became *Minimal Flower 3*, my first sculptural work – and an homage to Brent Collins, whose work had been so inspirational to me. It was first exhibited (in a stereolithography print at the 30 cm scale) at the Dayton, Ohio site of Intersculpt 2001. (See Figure 4, left. A smaller FDM model is shown in Figure 1, left.) A description of *Minimal Flower 3* much briefer than this was included in my 2003 essay "Optimal Geometry as Art" [5].

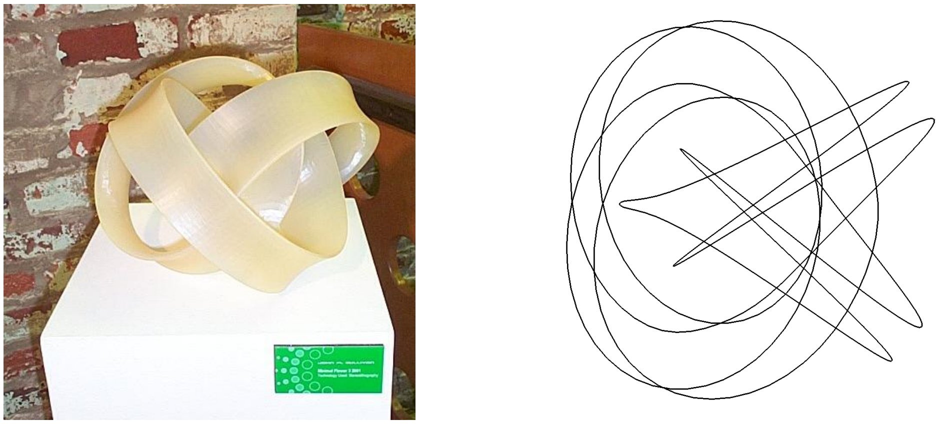

Figure 4: *Left: a 30 cm stereolithograph of* Minimal Flower 3 *on exhibit in Dayton in 2001. Right: the boundary curve for* Minimal Flower 4 *projected orthogonal to the axes of two of the helical pieces.*

3 Further Symmetries: *Minimal Flower 4*

When creating and naming *Minimal Flower 3*, I realized that it should be just one in a series of possible sculptures, minimal surfaces with $k22$ symmetry spanning appropriate knotted boundary curves for any k.

I have now tested this idea by designing *Minimal Flower 4*, the next in the series. The process was the same: start with a helical curve, use a four-fold rotation to get four symmetric copies, and tweak the angle of the axes and the cubic term in the equation of the helix to get them to join without corners. (One view of the resulting knotted boundary curve is shown in Figure 4, right.)

Now span this boundary curve with an initial surface sharing its 422 symmetry, and then minimize surface area to get a minimal surface – but work in hyperbolic space rather than euclidean space and be sure to enforce the symmetries. Finally, thicken the surface and blow a bit of air between the sheets to get the final sculptural model, as in Figure 1, right.

References

[1] Ken Brakke, *Surface Evolver*, `www.susqu.edu/brakke/evolver/`.

[2] Brent Collins, *Merging Paradigms*, Proceedings of *Bridges*, 1999, pp. 205–210.

[3] Brent Collins, *Visualization: From Biology to Culture*, Proceedings of *Bridges*, 2000, pp. 309–314.

[4] Carlo Séquin, *Volution's Evolution*, Proceedings of *ISAMA/Bridges*, 2003, Granada, pp. 13–24.

[5] John M. Sullivan, *Optimal Geometry as Art*, essay for Math Awareness Month (April 2003), reprinted in Proceedings of *ISAMA/Bridges*, 2003, Granada, pp. 529–532.

[6] Technische Universität Berlin, *3D Labor*, `www.tu-berlin.de/3dlabor/`.