Playing with the Möbius Band

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Abstract

If we are asked to visualize a Möbius band, we do not think first of its symmetry. However, if we make a model of a Möbius band with a computer program (for example, with Maple) and examine its boundary from different points of view, we get various interesting, symmetrical figures. A model of a Möbius band can be constructed by joining the ends of a strip (long rectangle) of paper with a single half-twist. It is interesting to observe how the resulting band transforms as we vary the ratio between the long and short sides of the rectangle. When will the surface intersect itself? We shall analyse these problems with multiply-twisted strips. The second part of this article deals with the connection between the Möbius band and frieze patterns.

Rosette Groups and the Möbius Band

A discrete group of congruence transformations of the plane without translation symmetries is called a rosette group. Rosette groups fall into two distinct families, according to whether they consist of rotations only (cyclic groups C_n), or also include reflections (dihedral groups D_n of order 2n, and C_n is a subgroup D_n). The boundary of a multiply-twisted band can be rendered as an attractive rosette in the plane.

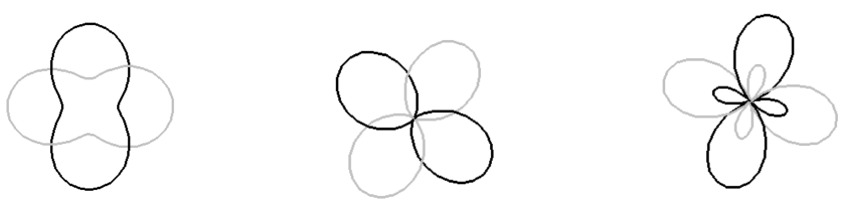

Figure 1 Boundary of the 4-twist Möbius band

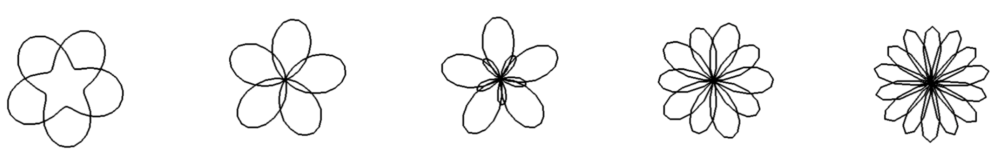

Figure 2 Boundary of the 5- and 7-twist band

Frieze Patterns and The Möbius Band

A frieze pattern is generated under the action of a discrete group of congruences, in which all translations are parallel to a single axis. A mathematical analysis reveals that there are seven different possible frieze patterns, in which one basic motif is repeated an infinite number of times. To illustrate frieze patterns we can use various designs such as alphabetics [2] and folk art design [4]. In this article I consider patterns whose motifs are projections of the boundary of a Möbius band and of a sometimes twisted band.

The Relationship Between Frieze Patterns and the Möbius Band

We now consider the relationship between the Möbius band, the cylinder, and the frieze groups. What does this mean graphically? We can make two-way joins of two opposite edges of a rectangle, as shown in Figure 3. In the first case we get a cylinder, in the second case a Möbius band.

Figure 3 Cylinder and Möbius band from a rectangle

Consider, for example, the pattern p112. Draw it into the first rectangle of Figure 3, then copy it by the arrows, as shown in Figure 4. Since the motif permits a halfturn, i.e., a twofold rotation about its midpoint, a second halfturn will also occur about a point between the two motifs, so that we obtain translation, according to the arrows.

The type p1a1 refers to the case where an asymmetric motif is inscribed into a Möbius rectangle and repeated using the oppositely-oriented arrows, i.e., by glide reflection (as indicated by the letter 'a' in the name). This latter transformation is nothing but a reflection followed by a translation along the reflection line (optionally applied in the opposite order). The composition characterizes the Möbius band. The glide reflection above can be produced also by a composition of a vertical reflection and of a halfturn, where the rotational centre does not lie on the mirror line. Then we can repeat the vertical reflection and halfturn along the horizontal line through the twofold centres as the pattern of type pma2 shows. One can then imagine that horizontal reflection or glide reflection in a pattern may involve the existence of a horizontal glide reflection in the pattern, and a Möbius band rectangle to the motif, with or without self-symmetry. Such a Möbius rectangle is possible with patterns p1m1, p1a1, pmm2, and pma2. With the other pattern types, only the cylinder rectangle can be used. Of course, two Möbius band rectangles together provide a cylinder.

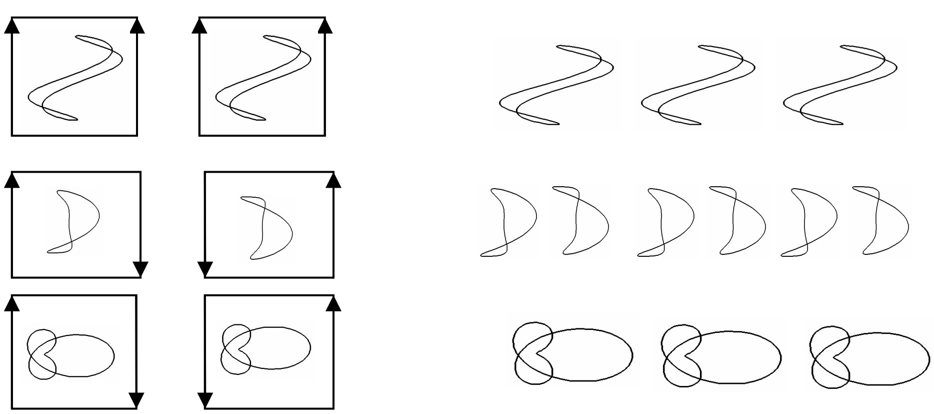

Figure 4 Cylinder, Möbius band and patterns p112, p1a1, p1m1, respectively

In some patterns, the motif itself is symmetrical. The name of the pattern type encodes this fact as well. We can choose a smaller asymmetric domain of the motif, in such a way that the symmetry operations already acting on this domain will produce the whole pattern. Such a smallest domain is called a fundamental domain (it is not unique, in general). This characterizes also the so-called quotient space or orbit space, since a fundamental domain (in its interior) contains only one point from each orbit. The first pattern is the single one, which has only one type of symmetry, namely a basic translation to generate the frieze group. The orbit space of this first pattern is just a cylinder.

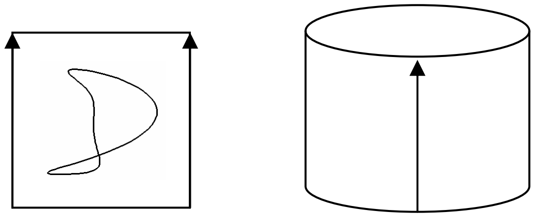

Figure 5 The quotient space or orbit space of the frieze group p111, the parallel arrows, glued together, yield a cylinder.

In the second pattern we assigned two halfturns, denoted by the green rhombs. In Figure 6 we glued together the image points on the edges of rectangle BCDE under the halfturns in such a way to become two cone surfaces to form an "open pillowcase".

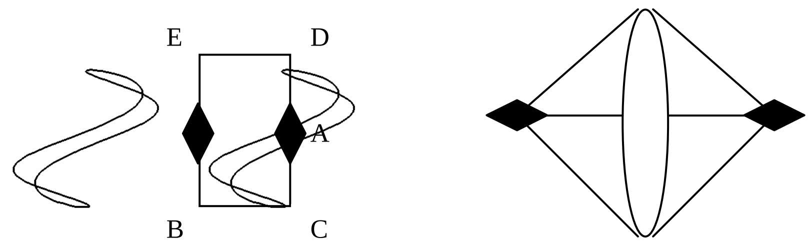

Figure 6 The quotient space of the frieze group p112

The third pattern contains a reflection in a horizontal mirror line. The surface will be a cylinder where the reflection mirror forms a boundary at the bottom.

Figure 7 The quotient space of the frieze group p1m1

The following pattern (Figure 8)—as mentioned above—is produced by one motif that is repeated by a glide reflection, yielding the Möbius band as the quotient space. We can notice analogous features in the last pattern (Figure 9). But interestingly, the motif of this pattern has a reflection symmetry in a vertical mirror, and a centre of twofold rotation as well.

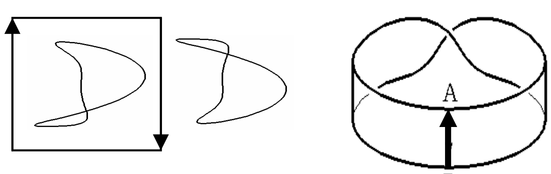

Figure 8 The quotient spaces of the frieze groups p1a1 and pma2

The fifth and sixth patterns contain only additional reflections, thus both quotient spaces are rectangles, though the horizontal reflection in pmm2 leads to an additional boundary segment.

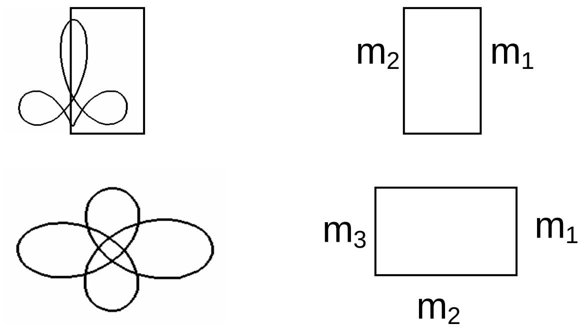

Figure 9 The quotient spaces of the frieze groups pm11 and pmm2

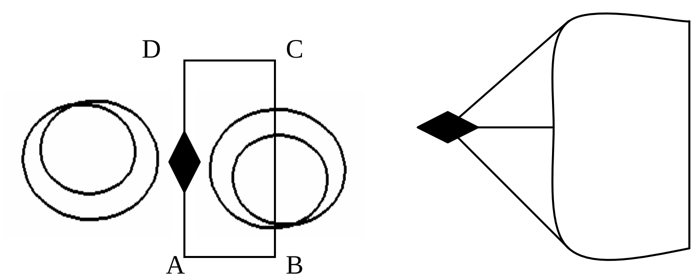

Figure 10 The quotient space of the frieze group pma2

Finally, we have a curiosity: Rotating the boundary of a Möbius band, a typical motif of Hungarian string-decorations appears.

Figure 11

References

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