Building the Schwarz D-Surface from Paper Tiles

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Periodic minimal surfaces are doubly curved and so problematic to create from paper, a material more amenable to developable surfaces. However, breaking the curvature into polygonal facets can visually approximate these surfaces. Furthermore, repeating and alternately inverting a single fundamental patch will tile periodic surfaces. This patch may be triangulated and unfolded into patterns for the modeler to print and fold into a number of non-planar tiles for constructing the surface. In the case of a lined periodic minimal surface, like the Schwarz D-Surface, the straight lines crisscrossing the surfaces define the boundaries of the fundamental patch as non-planar polygons. As demonstrated in this paper, such saddle polygons are relatively simple to fabricate and then to join into a representation of the surface.

Introduction

A hyperbolic paraboloid (hypar) module in the proper configuration offers intriguing possibilities for deploying saddles in space so as to tile periodic minimal surfaces. One especially elegant tiling scheme derives from the structural geometry presented by Peter Pearce in his book Structure in Nature Is a Strategy for Design. He proffers the case where either the 90º regular saddle hexagon or the 60º regular saddle rhombus can tile space to form the Schwarz D-surface (Figure 1). Pearce notes further that both non-planar, or saddle, polygons subdivide into identical hypar patches shaped like a kite. Consequently this patch serves as the common module, i.e., fundamental patch, for building Schwarz's surface.

In order to craft this module from paper the modeler must first triangulate the kite's surface and then unfold the triangular faces into a 2D pattern. Once printed and cut out with all of its fold lines scored, the pattern can fold into a 3D representation of the triangulated hypar kite. These kites then become the building units of the full Schwarz D-surface.

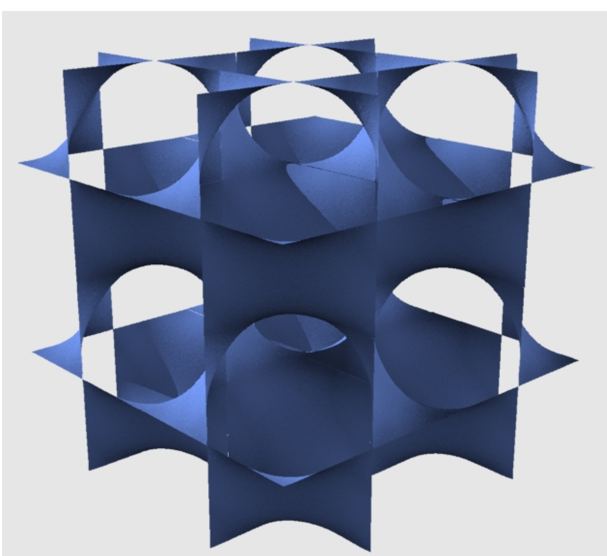


Figure 1. *Schwarz D-surface.*

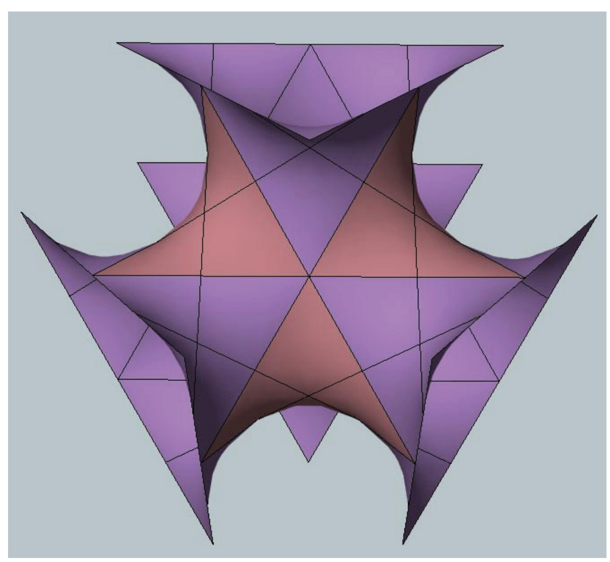


Figure 2. *Schwarz D-surface unit tiled from saddle hexagons.*

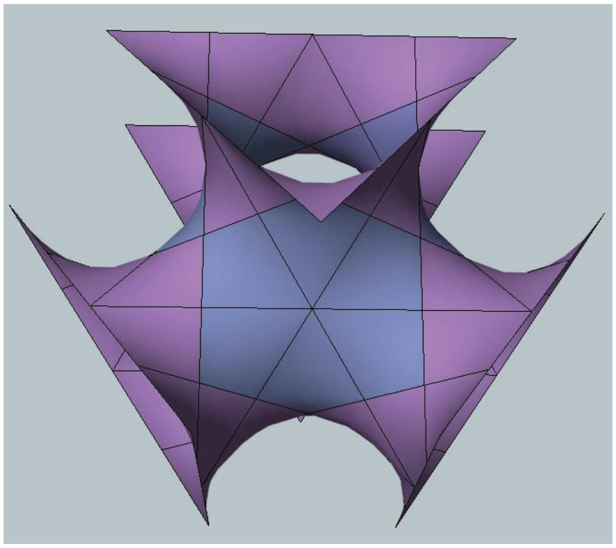


Figure 3. *Schwarz D-surface unit from saddle quadrilaterals*.

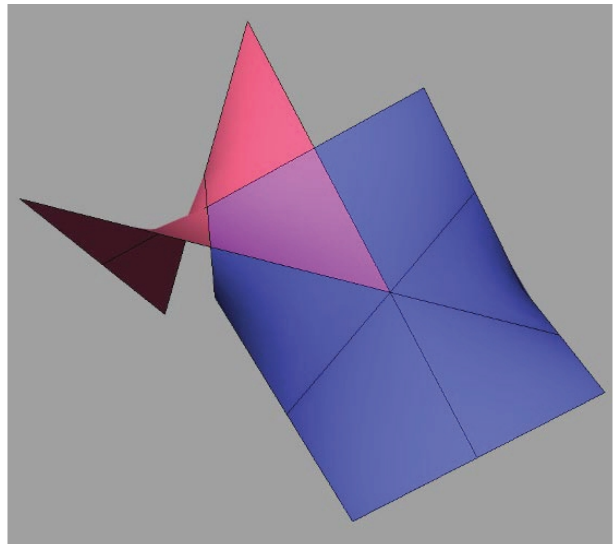
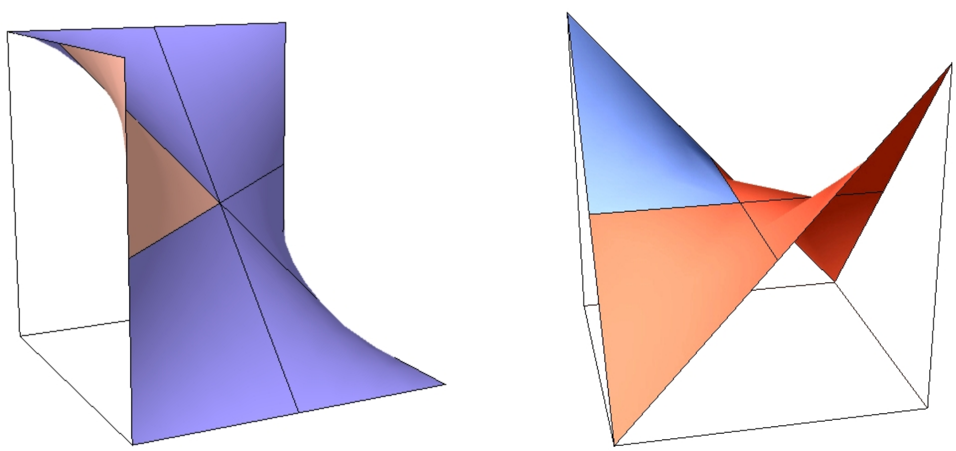


Figure 4. *Hypar kite is the fundamental patch of the D-surface.*

Saddle Polygons

Pearce tiled the Schwarz surface using either of two regular saddle polygons: the 60º saddle rhombus (Figure 2) and the 90º saddle hexagon (Figure 3). Pearce derived these, as well as a host of other saddle polygons, by extracting their edges from a dense grid of all possible cubic symmetries that he terms the Universal Node System, a grid matrix of cubes whose interiors and faces are traversed by their diagonals.



Figures 5 . *90° saddle hexagon and 60° saddle quadrilateral from the cube. Side faces of the right cube are cut on the diagonal.*

Traces of this grid remain on the Schwarz D-surface as the straight lines, which transcribe the surface while defining the edges of the tiling polygons. The intersections of these lines at the saddle point of each polygon mark out a "kite" saddle. Since this saddle tiles both the 60º rhombus and the 90º hexagon it also tiles the entire surface. Geometers variably term this kite as the "fundamental patch" or the "surface piece" of the Schwarz D-Surface. All triply periodic minimal surfaces feature an equivalent basic piece whose repetition builds the surface. None, however, are as simple as our kite.

A direct method for constructing these two saddles exploits the relationship each bears to the cube. A closed transit of six cube edges, for example, yields the 90º saddle hexagon. Similarly, a closed transit of diagonals on the side faces of the cube will define the sides of a 60° saddle rhombus. In each case half of the faces of the cube are removed to reveal edges of the respective polygons.

With the addition of the armature of mid-edge diagonals as shown in Figure 6 the stage is set for building the saddles from a circuit of hypar surface patches in the form of a kite, the fundamental patch discussed above. It turns out that the kites for both saddle polygons are congruent, allowing the 60º regular saddle rhombus and the 90º regular saddle hexagon to intersect by their identical kites as in Figure 4 above. With a little experimentation the possibilities for three-dimensional tiling of these saddle polygons become apparent, as in Figures 2 and 3 above.

Module Design

For the purposes of tiling shared hypar kite suffices. By triangulating the hypar kite it is possible to build physical representations of these surfaces from folded paper modules (Figure 6). The simplest such triangulation uses four triangles such that each triangle has as its base one side of the kite and as its apex the saddle point of the hypar (Figure 7).

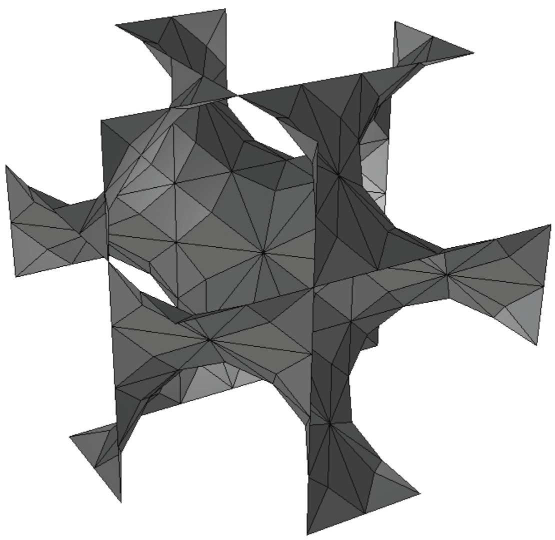


Figure 6. *Triangulated periodic surface.*

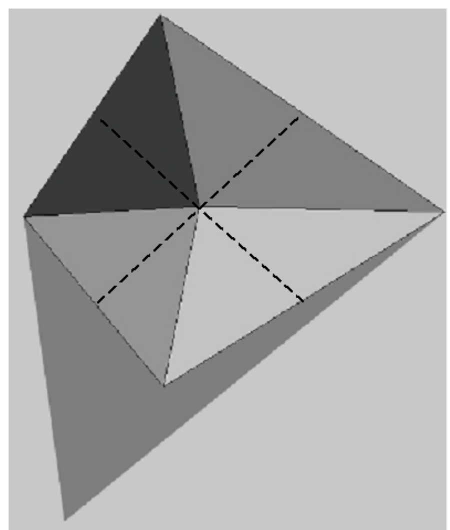


Figure 7. *Triangulated kite.*

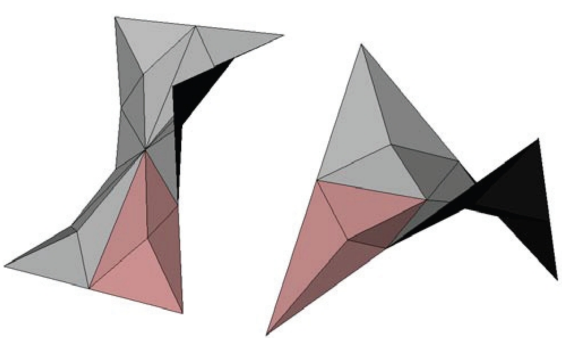


Figure 8. S*addle hexagon and rhombus triangulated from kites.*

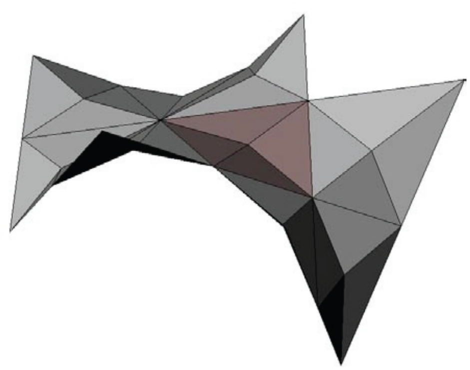


Figure 9. *Overlap of these polygons is a kite.*

Six kites orienting their curvature in alternating up-and-down directions and glued in a radial pattern by their long edges yields the 90º regular saddle hexagon, while four kites joined about a center by their short edges produces the 60º regular saddle rhombus (Figure 8). Thus the two polygons intersect by the same kite (Figure 9). Based on Figure 9 it is apparent that four 90º hexagons will tile around a common point, and that six 60º rhombuses will tile around a common point, to produce the same surface.

Figure 10a is a photograph of portion of the Schwarz D-surface constructed from these kites. Figure 10b reveals the same surface with one of its 90º hexagons outlined. The hexagon works as a kind of architrave at the juncture of the Schwarz surface's tubes. The 60º rhombus, on the other hand, wraps around the tubes (Figure 10c). These figures also reveal that at each juncture four tubes meet with four hexagonal architraves bridging between them in tetrahedral symmetry. The angles of intersection correspond to the intersecting edges of packed rhombic dodecahedrons.

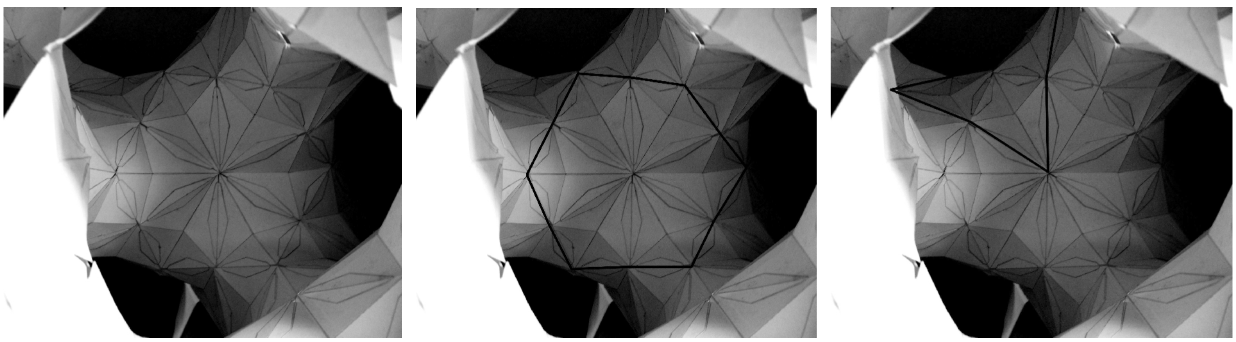


Figure 10a. *A view of the paper Schwarz surface through one of its hollows.* **Figure 10b.** *The 90º regular saddle hexagon on the Schwarz surface.* **Figure 10c.** *The 60º regular saddle rhombus on the Schwarz surface.*

Figure 11 diagrams the pattern for folding and assembling the kites. There are two patterns, one the mirror of the other. Trial and error proved that, since any two adjacent kites are flipped relative to one another, mirroring the patterns aided assembly. The kites hinge together with folded rhombuses keyed to the length of their edges. By mirroring the pattern the rhombuses fit to the dashed lines and all may align on the same side of the surface. This will generate a highly symmetric decorative pattern across the surface and transform what might have been a distractive joining into an enhancement of the surface.
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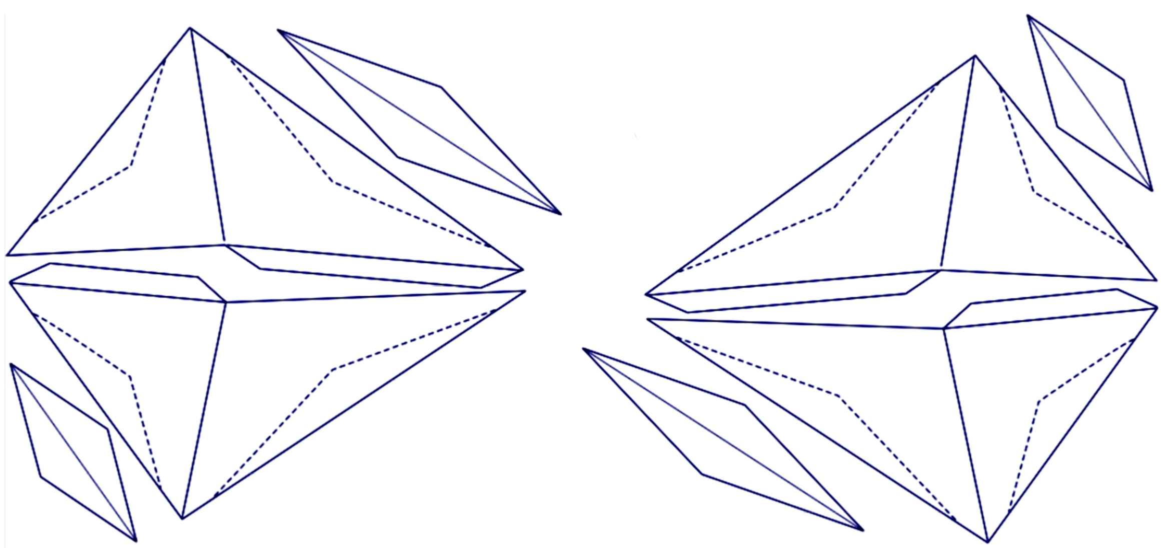

Figure 11. Patterns for the triangulated hypar kite.