

# Combinatorial Choreography

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## Abstract

Together with the workshop participants, we investigate various ways to express combinatorial structures through persons and/or physical objects in motion. For instance, the six permutations of three elements can be presented by three persons in a dance. Each person represents an element. They stand next to each other, and, five times, two appropriately selected neighbors swap places. Each swap can be artistically executed. Alternatively, the three persons can hold three physical objects to represent the three elements, and they exchange objects to present the six permutations.

This is a discovery workshop, where I will guide you through some of the combinatorial structures that I have investigated. Even simple combinatorial structures give rise to interesting mathematical questions when trying to present them through persons and physical objects. The design of a choreography can lead to surprising insights.

There will also be room to work on designing your own choreography for a combinatorial structure of your choice. Finally, we reflect on possible applications in education.

## Introduction

The International Mathematical Olympiad (IMO, see [5]) is the premier annual contest in mathematics for high school students from all over the world. For the contestants, the IMO typically takes more than a week, including an opening ceremony, two competition days, various social and cultural events, and an awards ceremony. The opening ceremony involves a flag parade, where all delegations present themselves on stage. During this parade, some delegations just bow, others toss items into the audience, and occasionally a little act is performed. During the opening ceremony of IMO 2009, I started thinking about a mathematically inspired act for the Dutch team at a future IMO. This workshop derives from those thoughts.

An IMO delegation consists of (at most) six contestants and two accompanying adults. Only the deputy leader is present in the flag parade, because the team leader—being informed about the contest problem—is quarantined. What can a team of six contestants act out on stage? An obvious idea is to let the contestants show something mathematical.

Classical mathematics is concerned with numbers and shapes. More modern entities studied in mathematics are so-called *structures*. A simple and well-known structure is the sequence, consisting of items having a specific linear order, with a designated first and last item. The IMO contestants could stand next to each other, in all possible orders. Gracefully transitioning from one order to another constitutes a dance.

After this introduction, the workshop participants will be exploring some structures, first by physically performing them in small groups, and later also on paper. We conclude by discussing possible applications in education. Workshop material and further details can be found at [9].

## Permutations

The participants break up in groups of four or five persons. The first assignment for each group is to find an elegant way to have three persons stand in line and take on all possible orders. Once a satisfactory ‘dance’ has been worked out for three persons, the next question is how to do this with four or five persons.

Each possible order to place  $N$  objects next to each other is called a *permutation* of the objects. There are  $N! = N \times (N - 1) \times \cdots \times 2 \times 1$  (pronounced as “ $N$  factorial”) permutations of  $N$  objects. Thus, zero and one object are not interesting, because there is just one order. With two objects (say,  $A$  and  $B$ ), there are two possible orders ( $AB$  and  $BA$ ), but the “dance” is trivial, consisting of just one swap of places. Note that such a swap still can be executed in many ways; we shall explore that during the workshop, but not here.

With more objects it becomes more interesting. The six permutations of three objects  $A$ ,  $B$ , and  $C$  are (in alphabetic order):  $ABC$ ,  $ACB$ ,  $BAC$ ,  $BCA$ ,  $CAB$ , and  $CBA$ . In a choreography, one is concerned with the transition from one state to the next state. Let us have a look at those transitions in the preceding sequence. The first, third, and fifth transition are just *neighbor swaps*, where two adjacent dancers (objects) trade places. The second and fourth transition are *cyclic rotations*: the person on the far right moves all the way to the left, or the person on the far left moves (back) to the right. When changing the order of the permutations to  $ABC$ ,  $ACB$ ,  $CAB$ ,  $CBA$ ,  $BCA$ ,  $BAC$ , all transitions are neighbor swaps, as illustrated below (permutations are shown vertically and each neighbor swap is marked by  $\times$ ):

$$\begin{array}{ccccc}
 A-A & C-C & B-B & & \\
 & \times & & \times & \\
 B & C & A & B & C & A \\
 & \times & & \times & & \times \\
 C & B-B & A-A & & C & 
 \end{array}$$

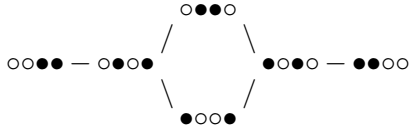
Note that one further neighbor swap of  $AB$  at the end would return all dancers to their starting positions. It is well known that, in general, the permutations of  $N$  objects can be ordered such that all transitions are neighbor swaps, e.g. via the so-called Steinhaus–Johnson–Trotter algorithm [6, 8, 11]. In fact, 17th-century English change ringers already applied this algorithm. Another algorithm, obtaining a different order for  $N \geq 4$ , is given in [2]. Both these algorithms are *recursive*, that is, the construction of an order of the permutations for  $N$  objects involves application of the same algorithm on fewer than  $N$  objects (if  $N$  is not too small). Especially the Steinhaus–Johnson–Trotter algorithm is easy to memorize and apply by dancers.

In the workshop, we will explore the case  $N = 4$  on paper. Figure 1 shows the graph whose nodes are the  $4! = 24$  permutations of four objects and whose edges connect permutations that differ by a single neighbor swap. This graph is known as a *permutohedron*. It is not so easy to find, in this rendering of the graph, a traversal that visits each permutation exactly once (see Fig. 3). Such a traversal is called a *Hamilton path* of the graph. In general, it is easy to check whether a given path on a graph is a Hamilton path, but it is hard (in fact, NP complete [3, pp.199–200]) to find out whether a graph admits a Hamilton path, and if it does, to exhibit such a path. The permutohedron of order 4 can be rendered in a nicer way, making it easier to find Hamilton paths. We do not show it here so as not to spoil the fun in the workshop; see [9] for details. Note that finding a Hamilton path can be considered a special case of the well-known *Travelling Salesman Problem* (TSP).

Unfortunately, the number of ways to place six items in a sequence is  $6! = 720$ , a number that is too large to handle conveniently in the ten seconds of stage time at an IMO. It would take twelve minutes to perform at one permutation per second. What other combinatorial structures can six dancers exhibit?

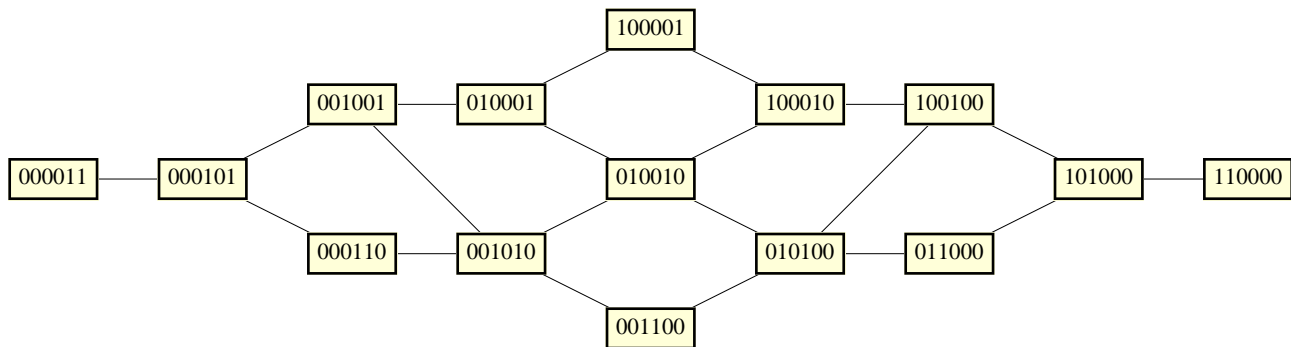
### Combinations

If some of the dancers are indistinguishable, then they can present structures that are known as *combinations*. For instance, if two dancers are dressed in black, and two in white, then there are  $\binom{4}{2} = \frac{4!}{2!(4-2)!} = 6$  possible sequences of these two colors:

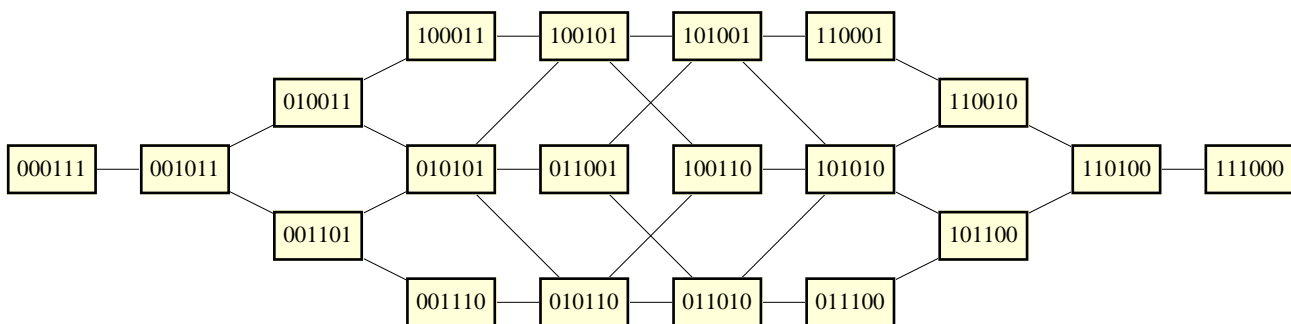


Each combination amounts to selecting a 2-element subset from a 4-element set: e.g., (the positions of) the two white dancers among the four dancers. In the diagram above, two combinations that differ by a neighbor swap have been joined by a line segment. From the resulting adjacency graph, it is obvious that it is not possible to order the combinations such that the transition between adjacent combinations is a neighbor swap. In mathematical terms, there exists no Hamilton path on this adjacency graph.

If one dancer is dressed in black and five in white, then they can stand next to each other in six ways. It is trivial to present these six combinations by neighbor swaps. If two dancers are dressed in black and four in white, they can line up in  $\binom{6}{2} = \frac{6 \times 5}{2 \times 1} = 15$  ways. Here is the adjacency graph (0 = white, 1 = black):



Unfortunately, there does not exist a Hamilton path in this graph (for an elegant proof, see Fig. 2). It turns out that also for 2-of-5, 2-of-7, and 3-of-7, such Hamilton paths do not exist. With three black and three white, the number of combinations is  $\binom{6}{3} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20$ , a number that should be doable in ten seconds. Try to find a Hamilton path in its adjacency graph:



In this case, it is possible to present these twenty combinations with neighbor swaps as transitions (see Fig. 4). Inspired by this topic, colleague Cor Hurkens recently found a general solution to the (non-)existence of Hamilton paths in the neighbor-swap graph of  $k$ -of- $n$  combinations. Later, we discovered that this was already known [1, 4]: a Hamilton path exists for  $k$ -of- $n$  combinations with neighbor swaps if and only if  $n$  is even and  $k$  is odd. In general, these dances are not so easy to memorize as those for permutations.

## Variations

A further generalization is possible by considering more than two colors, that can occur more than once. For instance, six dancers in three equally-colored pairs: two red, two white, two blue. There are  $\binom{6}{2\ 2\ 2} = \frac{6!}{2!2!2!} = 90$  such combinations. In this case, there exists no Hamilton path for neighbor swaps.

The restriction to neighbor swaps as means of transitioning from one state to the next could be relaxed. A ‘cool’ way to traverse all  $k$ -of- $n$  combinations (for any  $n$  and  $k$ ) is given in [7]: determine the shortest prefix that ends in 010 or 011 (or the entire sequence if such a prefix does not exist), and do a cyclic rotation by one position to the right. The cyclic rotation can also be accomplished by at most two simultaneous swaps (not necessarily of neighbors). This algorithm is doable in a practical setting.

Also worth consideration are *circular arrangements* of dancers doing neighbor swaps. Other candidates for a choreography are *de Bruijn sequences* [10].

## Education

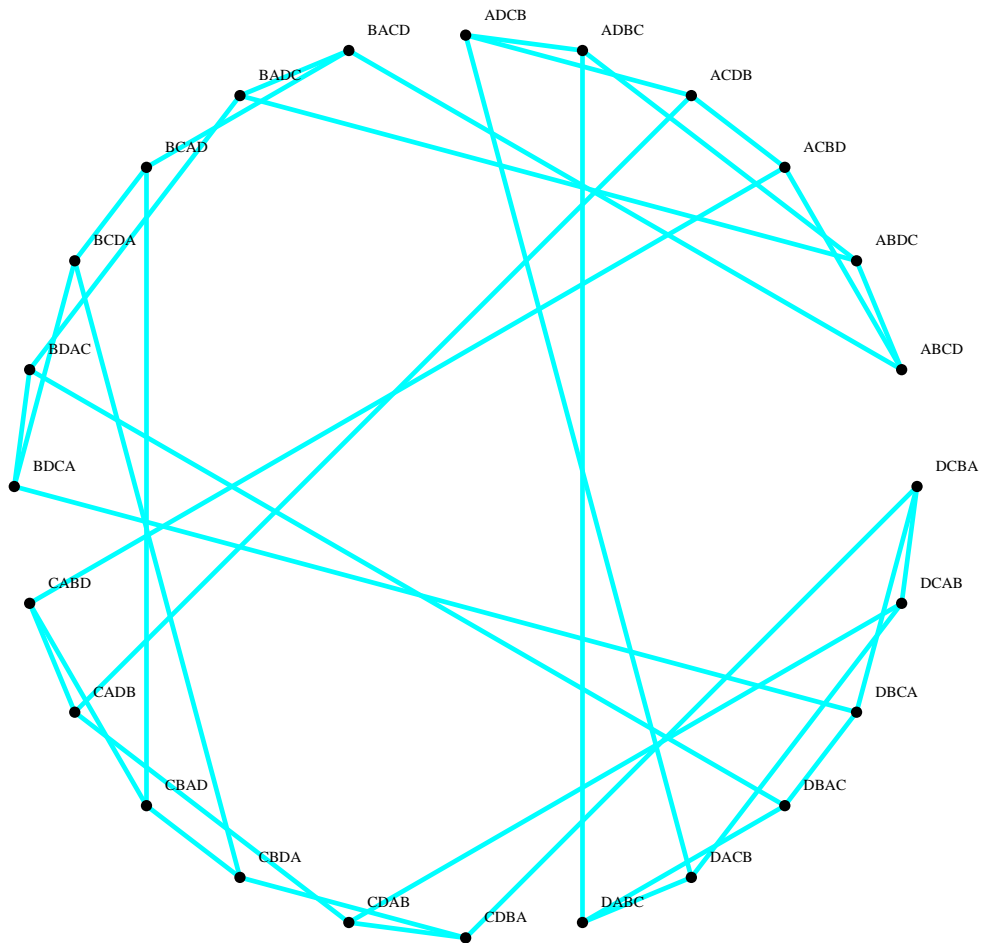
The search for choreographies for combinatorial structures gives rise to nice mathematical questions. This can be used in the class room to stimulate students to do their own mathematical explorations, including the formulation of relevant questions. Especially, but not only, young students can be stimulated by physically acting out small cases, appealing to kinesthetic learning. However, this should not be an excuse to avoid thinking about the mathematical problems.

## References

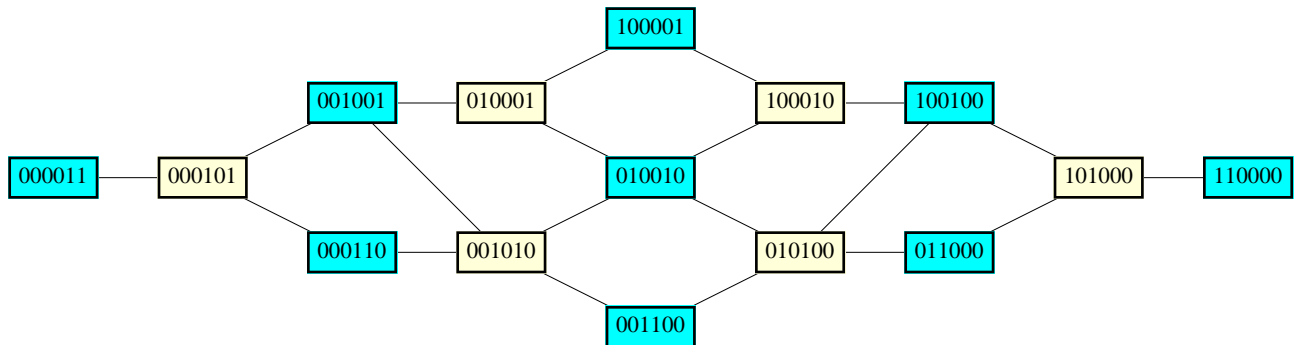
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- [11] Wikipedia. *Steinhaus–Johnson–Trotter algorithm* — *Wikipedia, The Free Encyclopedia*. [en.wikipedia.org/wiki/Steinhaus-Johnson-Trotter\\_algorithm](http://en.wikipedia.org/wiki/Steinhaus-Johnson-Trotter_algorithm) (accessed 29-Feb-2012).

## Appendix

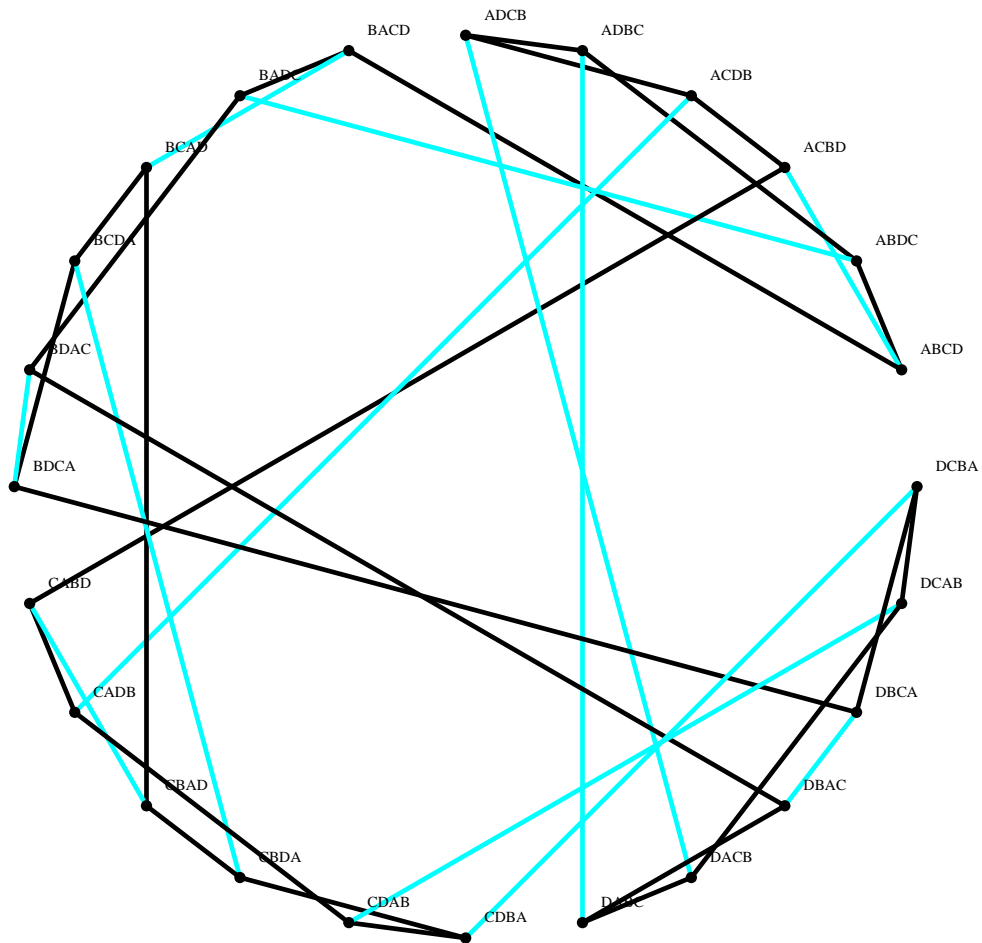
Some further illustrations for the workshop.



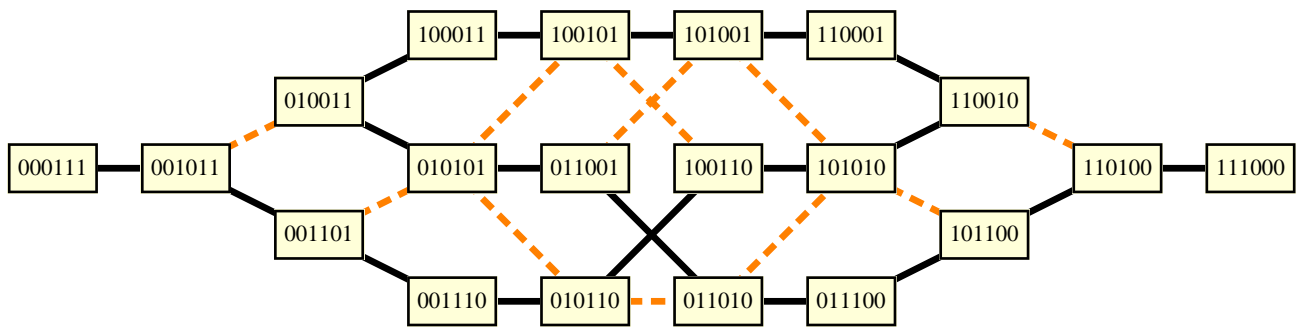
**Figure 1 :** *Adjacency graph on the 24 permutations of four objects. Also see [9]*



**Figure 2 :** *Adjacency graph for the fifteen 2-of-6 combinations, distinguishing ‘even’ and ‘odd’ combinations: on every path, the parities alternate; the difference in the number of ‘even’ and ‘odd’ combinations is 3; therefore, there exists no Hamilton path*



**Figure 3 :** Hamilton path (solid black) in adjacency graph on the 24 permutations of four objects.  
*N.B. There is a much nicer rendering, which will be shown in the workshop*



**Figure 4 :** Hamilton path (solid black) in adjacency graph on the twenty 3-of-6 combinations