Portraits of Groups in Three Dimensions

Jay Zimmerman
Mathematics Department
Towson University
8000 York Road
Towson, MD 21252, USA
E-mail: jzimmerman@towson.edu

And

Kevin Zimmerman
Towson University
8000 York Road
Towson, MD 21252, USA
E-mail: kzimme12@student.towson.edu

1. Abstract

This paper looks at the sculptures that result from representing a group G of order 8 as a group of three dimensional transformations. The action is realized as a quotient of a full quadrilateral group and so all cells have four edges. We use tetrahedra to represent the regions and each cell is adjacent to four others. The model of this 3-manifold in space must have a boundary. These sculptures are dynamic in the sense that each cell on this boundary may be moved to another part of the boundary to give a different sculpture.

2. Introduction

The first portrait of a group was given in Burnside [1]. Burnside used inversions in a circle to represent group elements. Each transformation is a transformation of the entire plane, but we can keep track of these transformations by looking at the image of a region ***E***. These images are always contained inside a circle which we identify with the Poincare Disk model of the hyperbolic plane. Each region is labeled with the composite of the transformations necessary to transform ***E*** to that region. Since inversion in a circle reverses the orientation of the plane, Burnside used a composition of two inversions for each element of the free group. The region ***E*** and any region obtained from ***E*** by an even number of inversions is a light color and any region obtained from ***E*** by an odd number of inversions is a dark color. A fundamental region for the group of transformations is the union of one light and one dark region. This pair of regions is labeled with the element of the free group that transforms the region ***E*** into that region. It has proven convenient to distort the two dimensional regions into polygons. This gives regions such as Figure 1, which is a polygonal region for a surface of genus five [2]. Each region on the boundary connects to another region on the boundary. These regions fold up to give a compact surface of a particular genus. Another nice property is that you can take a polygon from the boundary and relocate it elsewhere on the boundary. This leads to many nice plane designs. The purpose of this paper is to explore a 3-dimensional example and to examine the sculptures that might arise from such a polyhedral representation.

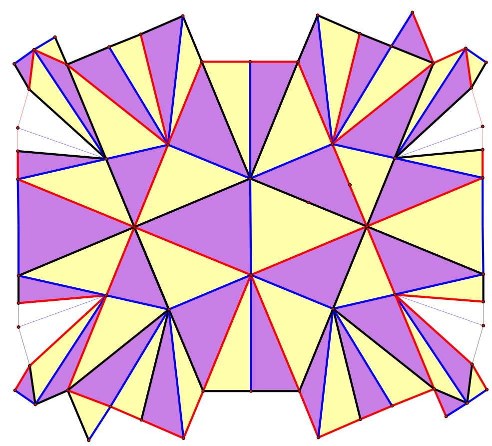

Figure 1 – Polygonal representation of G^+

3. Three Dimensional Polyhedral Regions

We propose to use tetrahedra for our first attempt at constructing a union of polyhedra that represent a group. Tetrahedral cells are fairly easy to construct and to move around to different configurations.

In order to facilitate this, our tetrahedra will have four colors on the faces. Each color represents the action of a different generator. The two dimensional polygonal regions in Figure 1 are not movable in this way, but it is easy to change a 2-D polygonal representation to make it look good. The big disadvantage of using tetrahedra is that the possible groups we can represent are very restricted. Each colored face of a tetrahedron represents an inversion transformation in a plane or sphere. As in the 2-dimensional case, the orientation preserving action of the generators of the group is represented by the composition of two such inversions. Unfortunately, five tetrahedra are all that can fit around a line. It follows that the order of the product of two inversions is at most two, since this would be represented by four tetrahedra around a line. It is easy to see that this forces the group to be elementary abelian. Since the group must have rank three, we see that G^+ is the elementary abelian group of order 8 and G is the elementary abelian group of order 16.

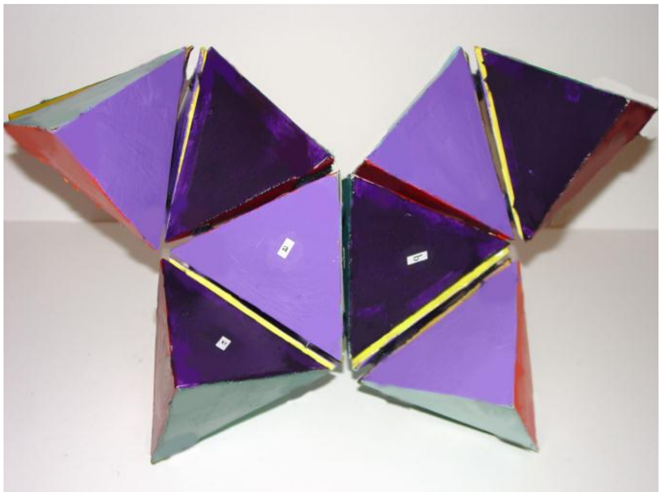

Figure 2 – Polyhedral representation of Z_2^3

The model that we have constructed is the set of all three dimensional regions of the elementary abelian group of order 8, Z_2^3. Using Magma, we can write the four inversions as permutations of 16 objects. These objects are the tetrahedra. The tetrahedra come in two varieties. One variety will be the original and have the same function as the yellow regions in Figure 1 and the other variety will represent the orientation-reversed cells, similar to the purple regions in Figure 1. The two varieties will have similar colors with the orientation-reversed cells a somewhat darker color. The solid that results from putting these tetrahedra together will have a boundary and each boundary cell will be connected to other cells on the boundary. We can see two configurations for this model in Figures 2 and 3. When this region is "folded", it results in a three dimensional manifold which can only be embedded in a higher dimensional space.

Figure 3 – Preliminary polyhedral representation of Z_2^3

Clearly, we cannot see the three-dimensional manifolds that result from this work. These manifolds can be embedded in higher dimensional Euclidean space and we may use the vertices of the cells and project them onto two and three-dimensional spaces. This might give very interesting artistic objects. A more interesting possibility would be to construct a movie of a "trip" through this 3-dimensional manifold and observe how it is folded on itself.

4. References

1. W. Burnside, *Theory of groups of finite order*, (Cambridge University Press 1911).
2. J. Zimmerman, A Group Portrait on a Surface of Genus Five, Conference Proceedings 2009, Bridges Banff: Mathematics, Music, Art, Architecture, Culture, Banff, Alberta, Canada, 259 – 264.