Patterned Triply Periodic Polyhedra

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Abstract

This paper discusses repeating patterns on infinite skew polyhedra, which are triply periodic polyhedra. We exhibit patterns on each of the three regular skew polyhedra. These patterns are each related to corresponding repeating patterns in the hyperbolic plane. This correspondence will be explained in the paper.

1. Introduction

A number of people, including M.C. Escher, created convex polyhedra with patterns on them. Three of Escher's polyhedra are shown on pages 246 and 295 of [11]. Later, Doris Schattschneider and Wallace Walker placed Escher patterns on convex polyhedra and on non-convex rings of polyhedra, called Kaleidocycles, that could be rotated and which are described in [12]. The goal of this paper is to start an investigation of drawing repeating patterns on a new kind of "canvas": infinite skew polyhedra — i.e. triply periodic polyhedra. Figure 1 shows a finite piece of such a pattern.

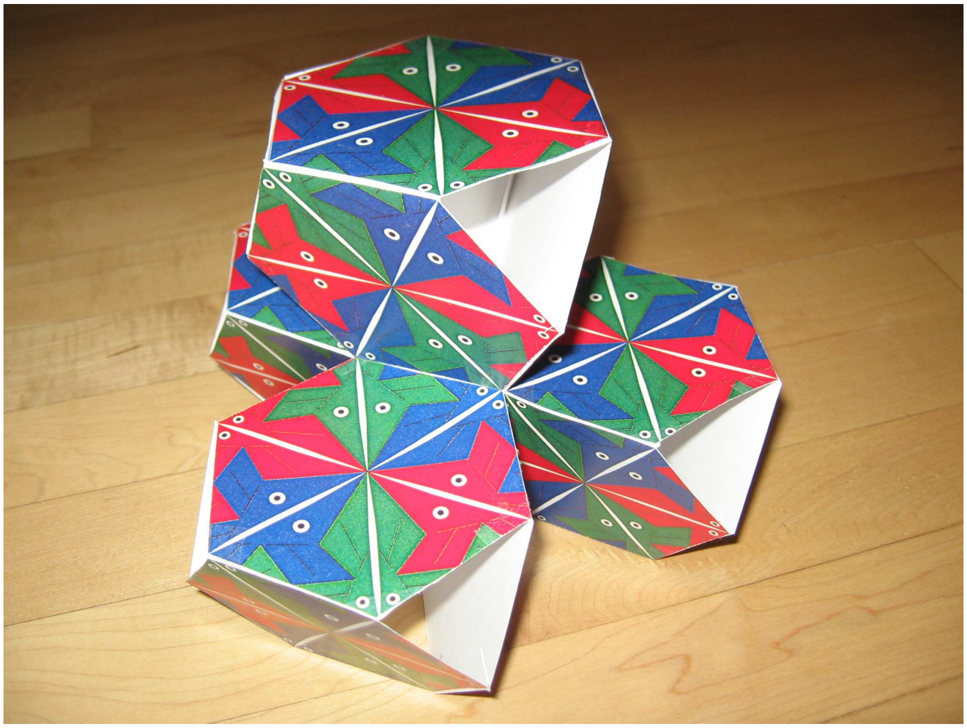

Figure 1: A pattern of fish on the tessellation $\{6, 6|3\}$.

We begin with a discussion of infinite skew polyhedra and show how they are related to tessellations of the hyperbolic plane. This relationship can also be applied to repeating patterns on those respective surfaces. Then we present patterns on each of the three regular triply periodic polyhedra. We also show a pattern on a non-regular triply periodic polyhedron. Finally, we indicate possible directions of further investigation.

2. Patterns, Hyperbolic Geometry, and Infinite Skew Polyhedra.

A *repeating pattern* is a pattern made up of congruent copies of a basic subpattern or *motif*. There can be repeating patterns on the Euclidean plane, hyperbolic plane, sphere, and polyhedra. Hyperbolic geometry is the least familiar of these surfaces, probably because (unlike the sphere) there is no smooth embedding of the hyperbolic plane into Euclidean 3-space [6], thus we must rely on models of hyperbolic geometry. We use the *Poincaré disk* model whose points are represented by Euclidean points within a bounding circle. Hyperbolic lines (which include diameters) are represented by (Euclidean) circular arcs orthogonal to the bounding circle. This model distorts Euclidean distances in such a way that equal hyperbolic distances correspond to ever-smaller Euclidean distances as figures approach the edge of the disk (the precise measure of distance is given in [9]). The Poincaré disk model was appealing to Escher and other artists since it represents the entire hyperbolic plane in a finite area and it is *conformal*, that is, the hyperbolic measure of an angle is the same as its Euclidean measure, so that motifs maintain their same approximate shape as they approach the bounding circle. For more on hyperbolic geometry, see [5].

A *regular tessellation* is a special kind of repeating pattern on the Euclidean plane, the sphere, or the hyperbolic plane. It is formed by regular p-sided polygons (equal edge lengths, equal vertex angles) or p-*gons*, q of which meet at each vertex, and is denoted by the Schläfli symbol $\{p, q\}$. If $(p-2)(q-2) > 4$, the tessellation is hyperbolic, otherwise it is Euclidean or spherical. Figure 2 shows the regular hyperbolic tessellation $\{4, 6\}$.

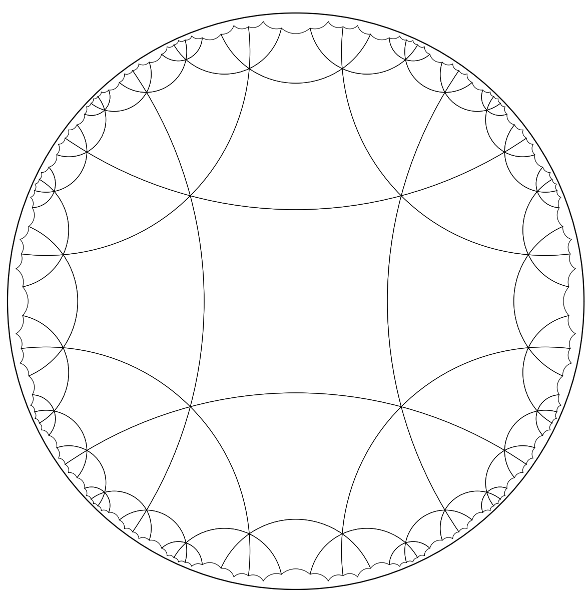

Figure 2: The $\{4,6\}$ tessellation.

We will consider polyhedra in Euclidean 3-space whose faces are all regular p-sided polygons (p-gons), and whose symmetry group is transitive on the vertices, so that there is a unique *vertex figure* — the polygon obtained by connecting midpoints of the edges incident at a vertex. If the polyhedron is convex, the vertex figure will be a planar polygon, otherwise it will be a skew polygon. Thus an *infinite skew polyhedron* is defined to have p-gon faces, to have a non-planar vertex figure (hence the name "skew"), and to repeat infinitely in three independent directions [7]. Since the vertex figure is non-planar, the interior angles of the polygons meeting at a vertex add up to more than 360 degrees; the difference between the angle sum and 360 degrees is called the *angular excess*. Infinite skew polyhedra have been called *hyperbolic tessellations* because they have positive angle excesses at their vertices. But we don't use this designation since it conflicts with our definition above. (They have also been named *polyhedral sponges* because they can be seen to divide space into polyhedral cells.)

A *flag* of a polyhedron is a triple: a vertex, an edge containing that vertex, and a polygon face containing that edge. A polyhedron is said to be *regular* if its symmetry group transforms any flag to any other flag; i.e. its symmetry group is transitive on flags. Thus *regular skew polyhedra* are special cases of infinite skew polyhedra whose symmetry groups are flag-transitive. There are exactly three of them, as proved by John Petrie and H.S.M. Coxeter in 1926 [2]. Their proof involved trigonometry as shown in a reprint of Coxeter's paper [4] (simple "counting" arguments showing that there are exactly five Platonic solids don't work since the polyhedra are not convex). Coxeter used the modified Schläfli symbol $\{p, q|n\}$ to denote them, indicating that there are q p-gons around each vertex and n-gonal holes. Figure 1 shows a fish pattern on $\{6, 6|3\}$. The other possibilities are $\{4, 6|4\}$ and $\{6, 4|4\}$, which we show in Figures 3 and 6 below.

A smooth surface has a *universal covering surface*: a simply connected surface with a covering map (projection) onto the original surface [8]. As examples, the Euclidean plane is the universal covering surface of the torus, and the sphere is the universal covering surface of the projective plane. The topological notion of a covering surface extends to Riemannian manifolds, which have metric properties, including curvature. If the original surface is negatively curved, its universal covering surface is also negatively curved and has the same large scale geometry as the hyperbolic plane.

A regular polyhedron has a universal covering surface that has a polyhedron-like structure, one of the tessellations $\{p, q\}$ of the sphere, Euclidean plane, or hyperbolic plane. For example, for regular skew polyhedra, the hyperbolic tessellation $\{p, q\}$ can be considered to be the "universal covering polyhedron" for $\{p, q|n\}$. Since regular skew polyhedra have positive angle excess, their universal covering polyhedra must be hyperbolic. We also extend the covering idea to repeating patterns on polyhedron: if the covering map from the universal covering polyhedron to the original polyhedron respects the pattern, we can say the patterned covering surface is the "universal covering pattern" of the original patterned polyhedron. As an example, Coxeter showed how to place 18 butterflies on a torus in pages 24–27 of [3]. The covering pattern is Escher's planar butterfly pattern Regular Division Drawing 70, page 172 of [11].

Infinite skew polyhedra are also related to triply periodic minimal surfaces (TPMS), since some TPMS surfaces are the (unique) minimal surfaces that span lines embedded in infinite skew polyhedra. Such TPMS's are thus intermediate between those polyhedra and their universal covering polyhedra. Alan Schoen has done extensive investigations into TPMS [13].

In the next three sections we show examples of patterns on the regular skew polyhedra and their associated hyperbolic "covering" patterns.

3. A Pattern on the $\{4, 6|4\}$ Polyhedron

The $\{4, 6|4\}$ polyhedron is the easiest to understand. It is based on the tessellation of 3-space by unit cubes. One way to visualize it is to index the cubes in each of the three directions by their integer coordinates and form a solid figure from only those cubes with one or three even coordinates (the complementary solid figure formed from cubes with zero or two even coordinates is congruent to it). The $\{4, 6|4\}$ polyhedron is the boundary of that solid figure. Figure 3 shows a pattern of angular fish on that polyhedron. These fish were inspired by Escher's first pattern in the hyperbolic plane, *Circle Limit I*. The colored backbones of the fish are embedded lines in the polyhedron. Those lines are also embedded lines in Schwarz's P surface, the corresponding triply periodic minimal surface shown in Figure 4. Figure 5 shows the corresponding pattern in the hyperbolic plane.

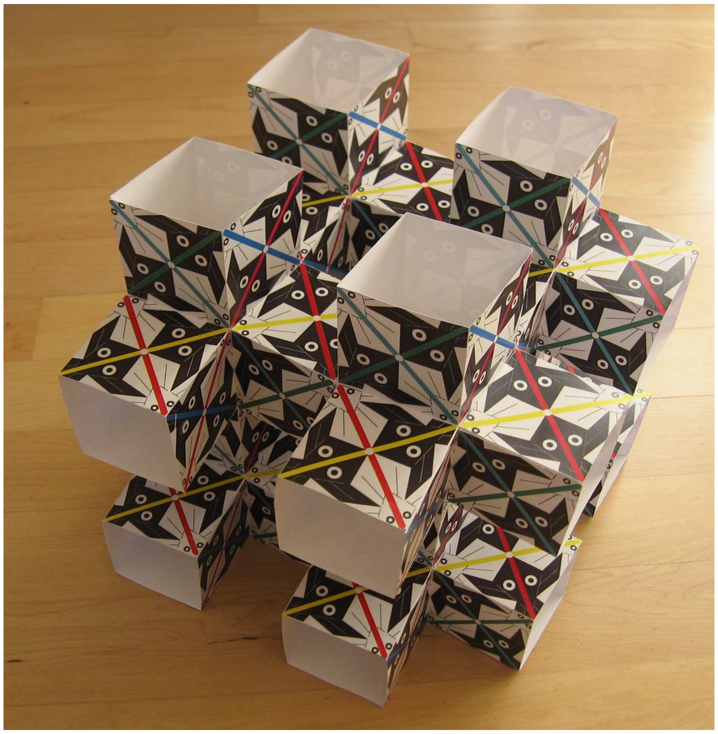

Figure 3: A pattern of angular fish on the $\{4, 6|4\}$ polyhedron.

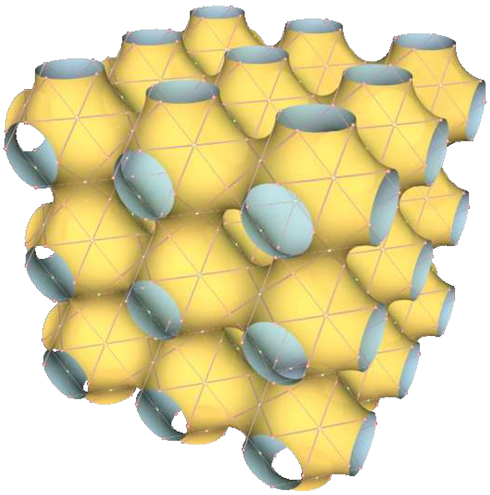

Figure 4: Schwarz's P surface showing embedded lines.

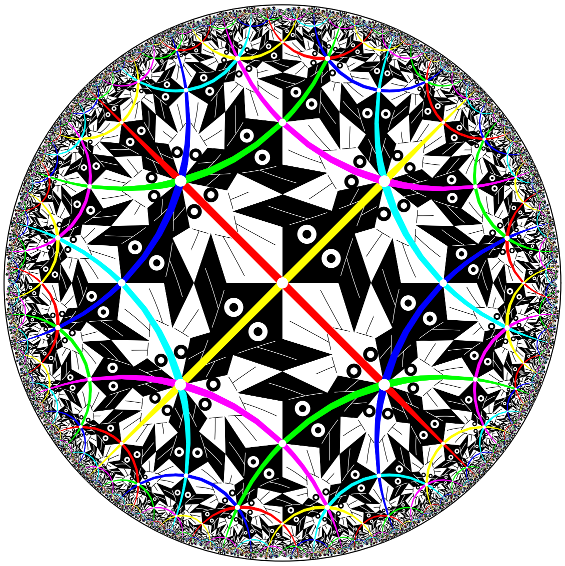

Figure 5: The hyperbolic pattern of fish corresponding to Figure 3.

4. A Pattern on the $\{6,4|4\}$ Polyhedron

The $\{6,4|4\}$ polyhedron is the dual of the $\{4,6|4\}$ polyhedron. The $\{4,6|4\}$ polyhedron is based on the bi-truncated, cubic, space-filling tessellation by truncated octahedra [1] (for details on truncated polyhedra in general see [14]). If we index rectangular lattice positions in 3-space as in the previous section, we can place one set of truncated octahedra at positions with all even coordinates, and a complementary set at positions with all odd coordinates such that all the truncated octahedra are congruent and fill space. The boundary between these two sets is the$\{6,4|4\}$ polyhedron. Figure 6 shows another pattern of angular fish on that polyhedron. As in the previous section, the backbones of the fish lie along lines embedded in the polyhedron. In fact the set of backbone lines is the same for both models. All fish along any backbone line are the same color. Figure 7 shows a top view of the polyhedron, and Figure 8 shows its universal covering pattern.

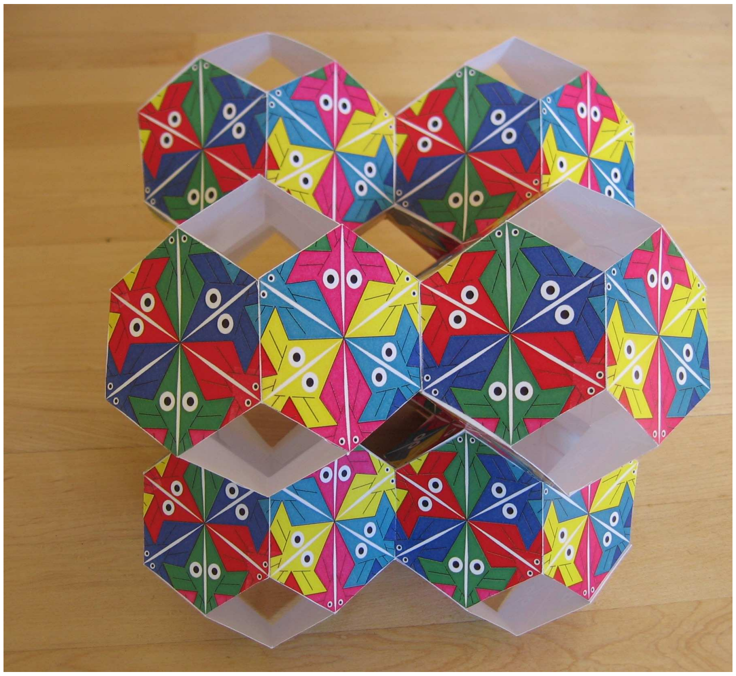

Figure 6: A pattern of angular fish on the $\{6,4|4\}$ polyhedron.

5. A Pattern on the $\{6,6|3\}$ Polyhedron

The $\{6,6|3\}$ polyhedron may be the most difficult to understand. It is formed from truncated tetrahedra with their triangular faces removed. Such “missing” triangular faces from four truncated tetrahedra are then placed in a tetrahedral arrangement (around a small invisible tetrahedron) [10]. A side view of a $\{6,6|3\}$ is shown in Figure 1. Figure 9 shows a top view looking down at one of the vertices (where six hexagons meet). Again, we placed a pattern of angular fish on this polyhedron. Figure 10 shows the corresponding universal covering pattern based on the $\{6,6\}$ tessellation.

All the fish along a backbone line in Figure 10 are the same color and swim the same direction. No two backbone lines of the same color intersect. In fact the pattern has 3-color symmetry (every symmetry of the polyhedron permutes the colors of the fish). The same comments also apply to the pattern of Figures 1 and 9. In the upward facing planes in Figure 1, the red fish swim lower right to upper left, the blue fish

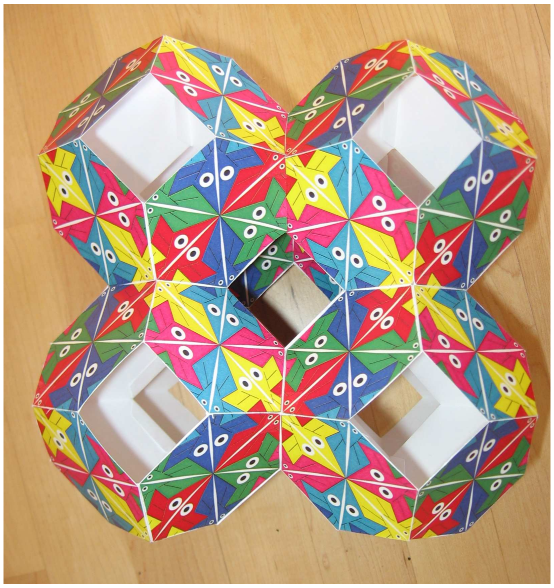

Figure 7: A top view of the pattern on the $\{6, 4|4\}$ polyhedron.

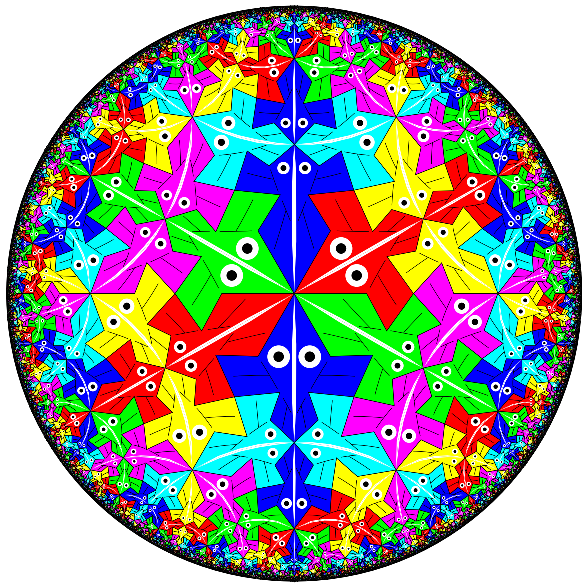

Figure 8: The hyperbolic pattern of fish corresponding to Figure 6.

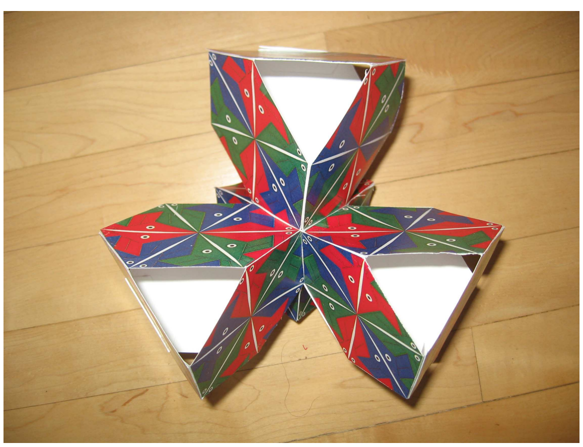

Figure 9: A top view of a pattern of fish on the $\{6, 6|3\}$ polyhedron shown in Figure 1.

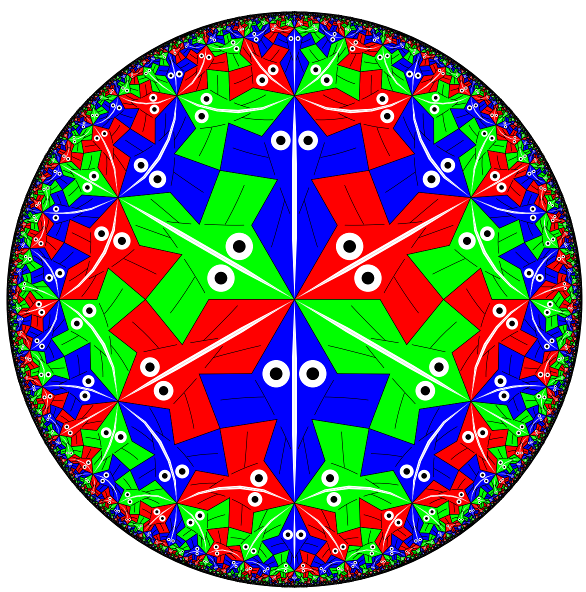

Figure 10: A pattern of fish based on the $\{6, 6\}$ tessellation.

swim lower left to upper right, and the green fish swim toward the viewer. In fact the backbone lines on the $\{6, 6|3\}$ polyhedron are embedded Euclidean lines.

6. A Pattern on a Non-regular Skew Polyhedron

Figure 11 shows a piece of a non-regular skew polyhedron. The whole polyhedron is made up of parts of regular octahedra of two types: "hub" octahedra and "strut" octahedra. Each hub octahedron has four strut octahedra placed on alternate faces of the hub, so four hub triangles are covered by struts and four are exposed. Each strut connects two hubs to opposite faces of the strut, which are covered by the hubs, leaving six exposed triangle faces. Thus eight equilateral triangles meet at each vertex, so we could designate this a $\{3, 8\}$ polyhedron. However it is not regular since there is no symmetry of the polyhedron that maps a hub triangle face to a strut triangle face (and vice versa). This polyhedron has diamond lattice symmetry and is closely related to the Schwarz D surface, which has embedded Euclidean lines [13]. Figure 12 shows a pattern of fish of four colors on the polyhedron. The rows of red, green, yellow fish each roughly follow the embedded lines of the Schwarz D surface; the blue fish make loops around the struts.

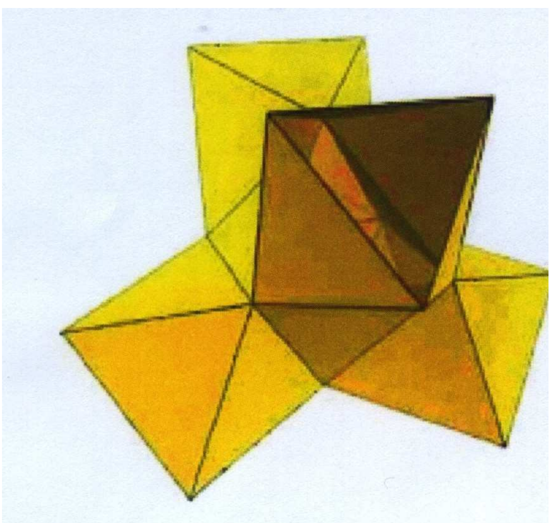

Figure 11: A piece of the $\{3, 8\}$ polyhedron.

Figure 12: A pattern of fish on the polyhedron.

7. Observations and Future Work

We have shown patterns on each of the regular skew polyhedra, but certainly many more patterns could be drawn on them. It is also possible to draw patterns on other infinite but non-regular skew polyhedra. In creating such patterns, it is desirable to take advantage of the combinatorics and any underlying geometry of the skew polyhedra, as was done with the patterns we have created. In summary, there are many more patterns on skew polyhedra to investigate.

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