From Two Dimensions to Four – and Back Again

Susan McBurney
Western Springs, IL 60558, USA
E-mail: smcburne@iit.edu

Abstract

Mathematical concepts of n-dimensional space can produce not only intriguing geometries, but also attractive ornamentation. This paper will trace the evolution of a 2-D figure to four dimensions and briefly illustrate two tools for going back again to 2D. When coupled with modern dynamic software packages, these concepts can lead to a whole new world of design possibilities.

The Emergence of New Methodologies

Many factors converged in the second half of the 19th century and the beginning of the 20th century that led to new ways of thinking, new ideas, and a search for untried methods of experimentation

In architecture, technological advances such as the development of cast iron and glass production opened the way for designers to develop new techniques and even new concepts that expanded upon these possibilities.

While some continued to rely on classic forms for inspiration, others such as Louis Sullivan, Frank Lloyd Wright, and lesser-known architect Claude Bragdon worked at developing their own styles, not only of architecture, but of integrated ornamentation as well. Geometric concepts played an increased role in style and methods and for Bragdon, became a prime source of inspiration. In his seminal book on ornamentation, "Projective Ornament" he stated, "Geometry and number are at the root of every kind of formal beauty."

At the same time parallel advances were taking place in many other disciplines. In mathematics concepts of dimensional spaces beyond three gained popularity. In particular, the idea of a 4-D space excited the imaginations of scientists, mathematicians, science-fiction writers, and the general public. This paper will briefly trace the progression from two dimensions to four and more importantly, from four back to two.

From Two Dimensions to Four

We obviously live in a 3-D world where no more than three dimensions can be physically constructed. This limitation leads some to question even the possibility of the existence of more than three. However, as Bragdon said in his book, "The concept of a *fourth dimension* is so simple that almost anyone can understand it… if only he will not limit his thought of that which is ***possible***….by his opinion of that which is ***practicable***." [1]

A simple example will illustrate the basic principle. Most readers of this paper will be familiar with the tesseract, or hypercube. Starting with a square, expand it in 3-D to form a cube. Now, extend this cube in each of four mutually perpendicular dimensions, and the result is a hypercube. We cannot build one in the 3-D world in which we exist, but we can represent one in 3-D and also in 2-D. See Fig. 3. This is a two-dimensional drawing of a three-dimensional representation of a four-dimensional object.

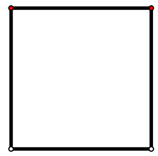

Figure 1: *2-D Square*

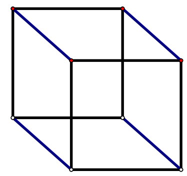

Figure 2: *3-D Cube*

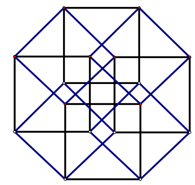

Figure 3: *4-D Hypercube*

Bragdon continues his argument with this quote: "It is not reason, but experience, that balks at the idea of four mutually perpendicular directions. Grant, therefore, *if only for the sake of intellectual adventure*, that there is a direction towards which we cannot point, at right angles to every one of the so-called three dimensions of space, and then see where we are able to come out." [1]

It is this idea of intellectual adventure that frees the mind to imagine other possibilities and leads to concepts which not only continue to obey well-know rules and structure, but also provide applications which are useful in lower dimensions. A very brief overview of "where we are able to come out" will follow.

Four Dimensional Space

In the two-dimensional plane there are an infinite number of regular polygons, figures with equal-length straight-line sides and equal angles. The equilateral triangle, the square, the pentagon, hexagon, and octagon are the most common and most easily recognized. In 3-D space polygons can be arranged along adjoining edges to produce figures called polyhedrons, The regular polyhedrons with equal angles, sides, and faces are five in number, known as the Platonic solids--the tetrahedron, the cube, the octahedron, dodecahedron and icosahedron.

Moving to four dimensional space, the solids formed by combining 3-D polyhedrons were known as polyhedroids to Bragdon, but are now referred to as polychora. The first three regular polychora are the 5-cell**,** composed of five tetrahedra, the tesseract (hypercube) which consists of eight cubes arranged so as to share edges, and the 16-cell, composed of sixteen tetrahedra arranged in four-space. The 24-, 120- and 600-cells complete the list of convex regular 4-D solids, but are not shown here for obvious reasons. The number of vertices, edges, face, and cells (component pieces) are summarized in Fig. 4.

Regular Convex 4-D Solid	V	E	F	Cells
5-cell	5	10	10	5 tetrahedra
Tesseract (Hypercube)	16	32	24	8 Cubes
16-cell	8	24	32	16 tetrahedra
24-cell	24	96	96	24 octahedra
120-cell	600	1200	720	120 dodecahedra
600-cell	120	720	1200	600 tetrahedra

Figure 4: *Regular 4-D convex solids*

From Four Dimensions to Two

The mathematician has a number of tools available for representing higher dimension objects in a lower-dimensional space. Two will be utilized here. The first is projective geometry, the familiar method which allows us to represent a cube on a 2-D drawing with no loss of understanding [Fig. 2]. The orientation of the object can be varied as well.

Another method is that of "unfolding" an object into a lower dimension. For example, a cube can be sliced along its edges and laid flat in the plane where it becomes six adjacent squares. Similarly, a tesseract, or hypercube can be sliced along its *planes* and unfolded into its cubic components in three-space. [Fig. 5]. These can then be sliced again along their *edges* and flattened onto the plane.

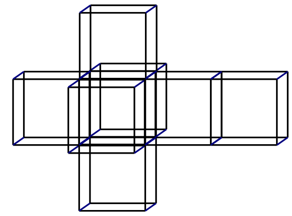

Figure 5: *Tesseract unfolded*

Designing in 4-D

Just the four-dimensional solids mentioned so far hold a wealth of decorative possibilities. Bragdon, in fact, created an entire ornamental vocabulary using only variations on the first three regular 4-D solids, by manipulating them in various ways in the plane to develop a significant body of ornament. His methods are available to present-day artists as well, plus the development of modern graphic software programs greatly increases the ease of exploration. A number of examples will be shown.

The hypercube pictured in Fig. 6a, when drawn in Geometers Sketchpad, can be stretched while maintaining its structural integrity by simply dragging the red line downward. The corresponding side lines extend automatically to give the figure in Fig. 6b. Four of these rotated about the bottom right point lead to the decorative figure in Fig. 6c. The scale decreases from left to right.

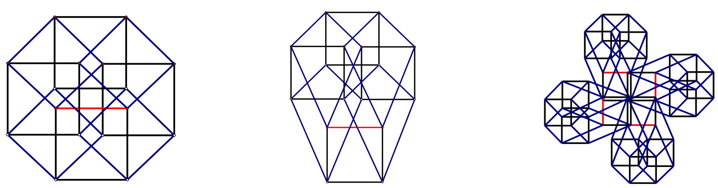

Figure 6: a b c

Similar manipulation easily leads to more designs, as shown in Fig. 7.

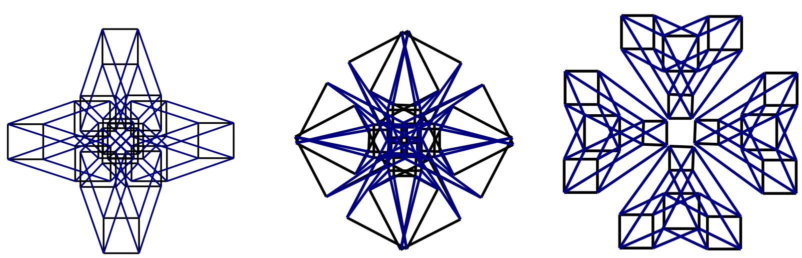

Figure 7: *Hypercube manipulation*

Four-dimensional prisms offer another rich source of design. In 3-D a regular prism is a polyhedron with two congruent parallel polygonal faces and the remaining faces are rectangles. Even the relatively simple 4-D example shown in Fig. 8 can lead to attractive motifs. Moving the central rotation point allows overlapping of the basic design to add complexity.

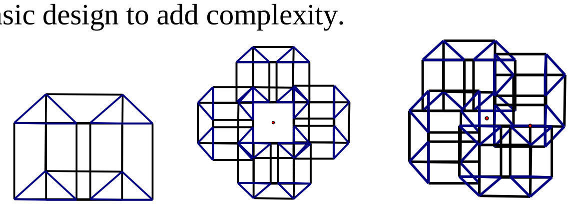

Figure 8: *4-D Prisms, rotated and overlapped*

Reflection across a line also leads to decorative ornament [Fig. 9].

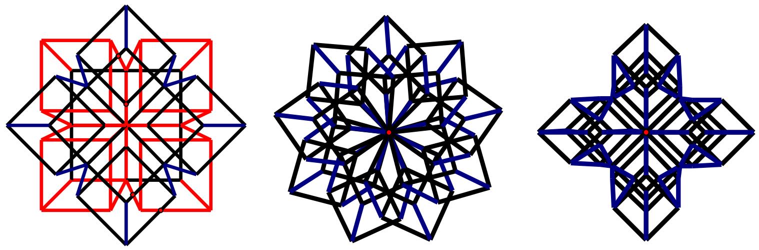

Figure 9: *Reflection of cubes*

Dilation is another tool in Geometers Sketchpad that can be used to advantage. Here the basic design has been reduced about the central point to give a smaller copy that adds interest to the interior. Added to these tools are those of an embellishment program as illustrated in Fig. 10.

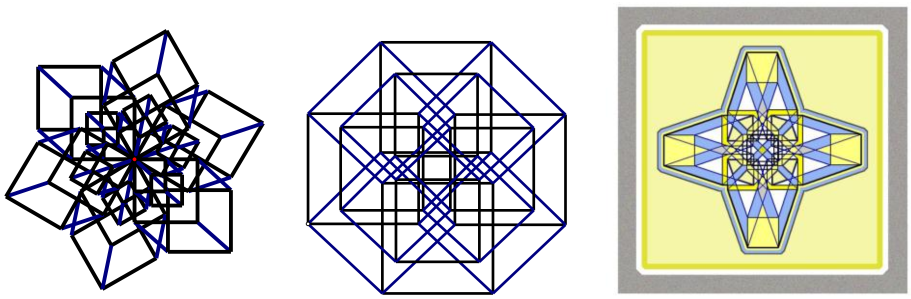

Figure 10: *Dilation and embellishment*

And of course repetition adds another whole new dimension (figuratively) to the design possibilities.

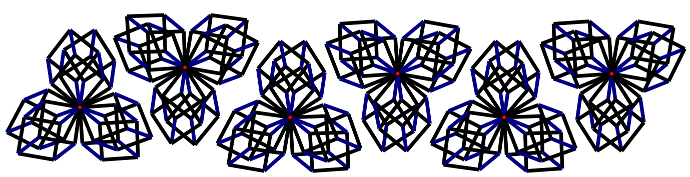

Figure 11: *Repetition of a 4-D motif*

Hopefully this brief exploration will give the reader some suggestion of the possibilities just waiting to be explored in four dimensions, as well as an appreciation for the rewards of opening one's mind to "intellectual adventure".

References

[1] Claude Bragdon, *Projective Ornament*, The Manas Press, 1915 Reprint by Kessinger Publishing
[2] en.wikipedia.org/wiki/Convex_regular_4-polytope
[3] Weisstein, Eric W. "Prism." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/Prism.html
[4] H.S.M. Coxeter, *Regular Polytopes*, 1963 (Dover, 1973)
[5] Susan McBurney, *The Projective Ornament of Claude Bragdon*, Joint Mathematics Meeting, Boston, MA, January 2012