My Parade of Algorithmic Mathematical Art

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Abstract

Animations are discussed for a range of geometric dissections. These include a variety of twist-hinged dissections, some swing-hinged dissections, dissections of squares and cubes corresponding to a certain integer identities, folding dissections of multi-level regular polygons, and folding dissections of multi-level polyominoes. Artistic issues pertaining to these dissections and animations are discussed, such as symmetry, synchrony, similarity, minimality, and color patterns. The resulting parade can be found at: http://www.cs.purdue.edu/homes/gnf/parade.html

1. Introduction

A geometric dissection is a cutting of a geometric figure into pieces that we can rearrange to form another figure. Such visual demonstrations of the equivalence of area span from the ancient Greeks through current-day advances on the world-wide web. During the past century, the emphasis has been on minimizing the number of pieces for any given dissection, which has catalyzed some remarkably beautiful creations. This was the subject of my first book [5], in 1997.

As dissection methods have become more sophisticated, attention has also focused on special properties, such as all pieces of a dissection being connected by hinges. *Swing hinges* allow the pieces to form one figure when they are swung one way on the hinges, and to form the other figure when swung the other way on the hinges. My second book [7], in 2002, focused on swing-hinged dissections, along with some "twist-hinged" dissections. A *twist hinge* allows one piece to be twisted around relative to the other. My third book [8], in 2006, explored *folding hinges*, which allow pieces to be folded one on top of another.

With talks based on my second book, I found it useful to demonstrate actual physical models. For my third book, the publisher included a CD with video-tape of me demonstrating various folding models. That led me to produce computer-generated animations for talks on material discovered after my third book. In the summer of 2011, I packaged those animations into a "parade of algorithmic mathematical art," which became part of a 3-person exhibition, "Art Comes to Lawson." The exhibition was installed in our Lawson Building of Computer Science to commemorate the building's fifth anniversary.

The intention was to take almost 50 single animations and videos and string them together into one large file that would then run continuously on an appropriate display device. The display device would sit inside a display cabinet that could not be opened by people viewing the show. Thus the animations would march past the viewers one after another, just as in a parade. That motivated the title: a parade of algorithmic mathematical art. Well, Alexander Calder had his circus [18], and I have my parade!

What makes this all *algorithmic mathematical art*? First, there is the application of elegant dissection methods to symmetrical figures such as regular polygons and stars. Second, there is the design of decorative objects such as a table whose top swings from an equilateral triangle to a square, or garden benches that reconfigure around a tree or a lamp post. Third, there is the way individual motions are smoothly executed via a sine function. Ideally, these motions reveal symmetry and other mathematical structure.

Algorithms are also present in these examples, both in the creation of smooth motion and in the use of dissection methods on certain infinite families of figures. This feature of the work is perhaps more hidden, because of the primary reliance on animation for its display. However, if you pay close attention to the animations you will at least get some hint of the underlying algorithms. In any event, relax and enjoy the nifty motion.

Constraints on the display device dictated that the parade, some 20 minutes in length, be partitioned into five different segments. This article features this same format, with each section focusing on one of the segments. The conference talk will bracket each segment with discussion.
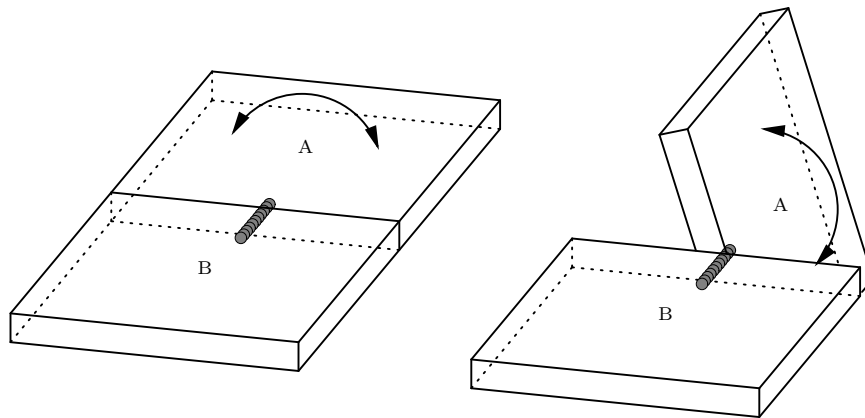
2. Twist-hinged dissections

Figure 1: A twist hinge for pieces A and B

It is surprising that we can use twist hinges (as shown in Figure 1) to transform one geometric figure to another [6, 9, 11]. It is even more surprising that we can take a swing-hinged dissection of an equilateral triangle to a square and convert it into the twist-hinged dissection of Figure 2(b). Just steal isosceles triangles whose apexes are adjacent to each swing hinge, as shown with the dashed edges in Figure 2(a), and glue them together!
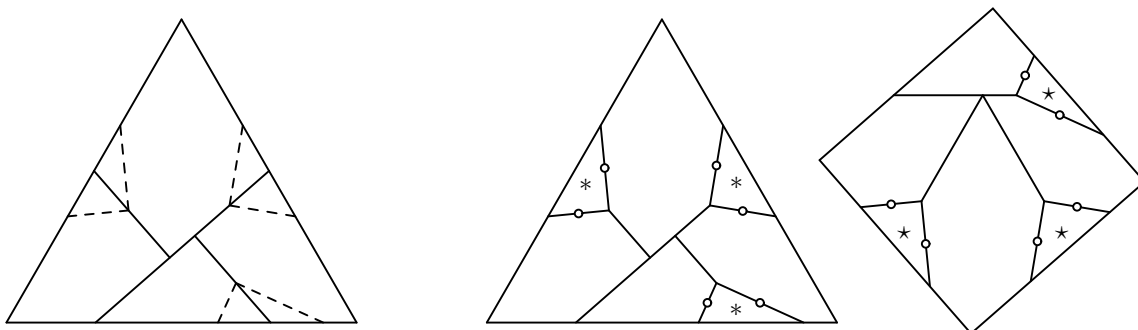

Figure 2: (a) Isosceles triangles   (b) Resulting twist-hinged dissection of an equilateral triangle to a square

We often get the raw material for this operation by forming a strip for one figure and overlaying it with the strip of another figure. An example is the overlaying of a strip for the {12/2} star with a strip of squares,

as shown in Figure 3(b). To get the strip element for the {12/2} star, we sliced six of the (fat) points off the star, leaving a regular hexagon, which we sliced in half, as shown in Figure 3(a). The resulting eight pieces can be arranged into the strip element. Special-case tricks are also sometimes used. One infinite family of dissections that we identify is of a (2k)-sided regular polygon to a k-sided regular polygon.
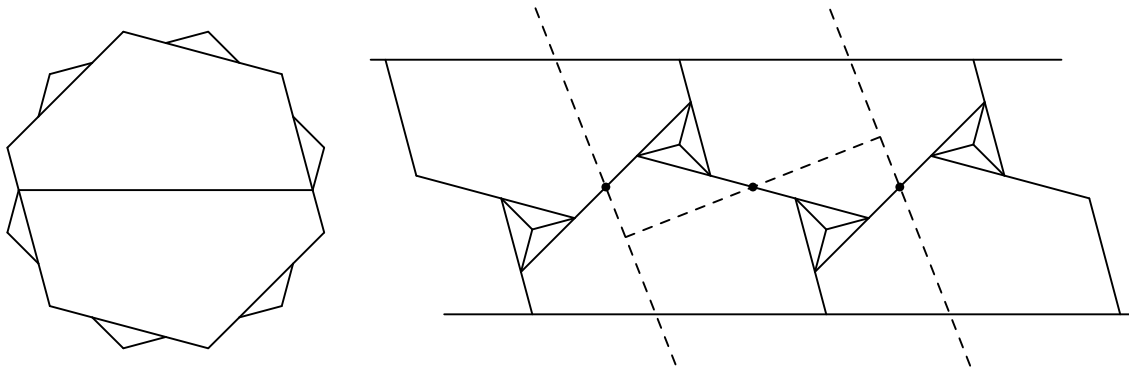

Figure 3: (a) Slicing a {12/2} star   (b) Overlaying strips for the dissection of a {12/2} star and a square

In the corresponding section we give animations of our twist-hinged dissections of the following figures to a square: an equilateral triangle, an {8/3} star, a regular decagon, a {6/2} star, a regular heptagon, a regular hexagon, a {12/2} star, and a regular octagon. We also give animations of our twist-hinged dissections of the following figures to an equilateral triangle: a regular octagon, a regular decagon, a regular pentagon, and a regular hexagon. Finally we give an animation of our dissection of an {8/3} star to a regular hexagon. In the parade, you can spot the "synchronized twisting" in the animation of the {12/2} star to the square.
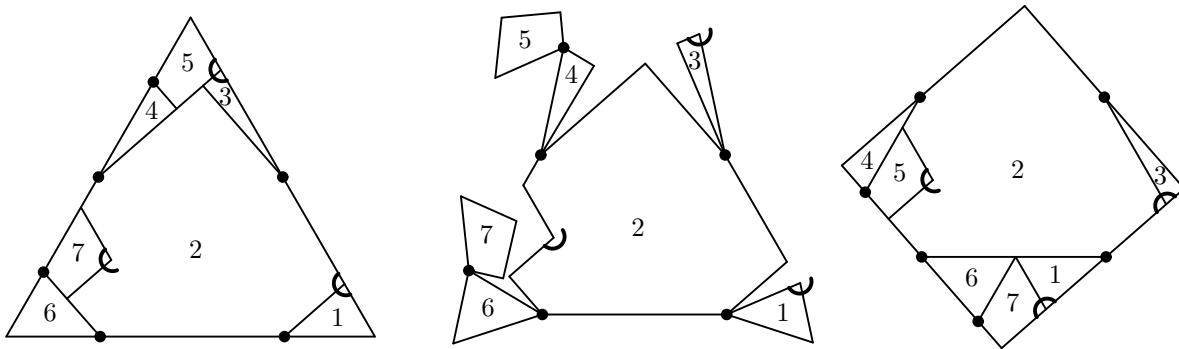
3. Transforming tables and benches

Figure 4: A different hinged dissection of an equilateral triangle to a square, with hooks.

In this section we consider geometric dissections that have demanded realization as physical models. A century ago the renowned puzzlist Henry Dudeney described a hinged model of an equilateral triangle to a square [3]. Inspired by this construction, the mathematician Howard Eves [4, pp. 37–38] described a set of four connected tables which he had built that would swing around to form either a square or a triangular top. The relative instability of those tables prompted me to design a pedestal table with much greater stability [12]. The dark dots represent the hinges. Piece 2 in Figure 4, which is supported by the pedestal leg, contains

75% of the area of the table top. The notch on the left in piece 2 allows it to accommodate either piece 5 (as in the square) or piece 7 (as in the triangle). Note the hooks that hold pieces in place in either the triangle or the square. Also animated is a version in which the notch is placed on the right rather than the left.

If we focus on polygonal rings, we can design twist-hinged dissections of rings to polygons and to other rings. These have a natural application in the design of reconfigurable garden benches [10]. They can result either by using the geometric structure of various polygons, or by converting the trapezoids of one polygonal ring to trapezoids of a different ring. Each trapezoid gets converted to a trapezoid of a different shape, but of the same height. We can see the conversion to trapezoids of two different shapes in Figure 5. The five trapezoids to the left of the dark line segments in the dodecagonal ring are converted to the five trapezoids in the pentagonal ring, while the seven trapezoids to the right of the dark line segments in the dodecagonal ring are converted to the seven trapezoids in the heptagonal ring.
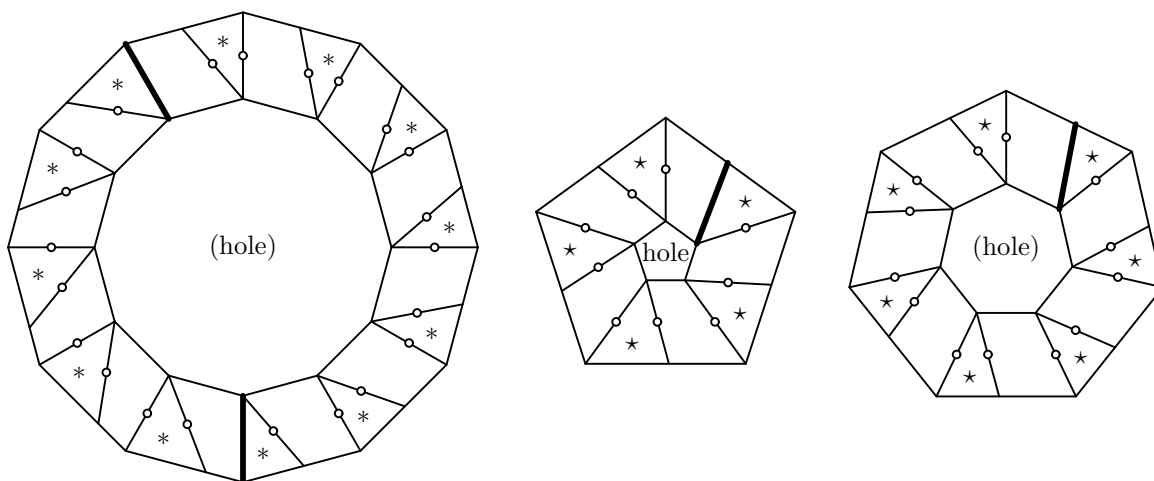

Figure 5: A twist-hinged dissection of a dodecagonal ring to a pentagonal and a heptagonal rings

In this section of the talk we animate: a decagonal bench that reconfigures into two pentagonal antiring benches, a decagonal ring bench to two pentagonal benches, a dodecagonal ring bench to three square benches, two dodecagonal ring benches to three octagonal ring benches, a dodecagonal ring bench to a "square" ring bench, and a dodecagonal ring bench to a pentagonal ring bench and a heptagonal ring bench.

4. Squares and cubes

Next we enjoy several infinite families of dissections based on identities involving sums of squares of integers, and also sums of cubes of integers [13, 15]. The technique for identifying the identities of squares was discovered by Georges Dostor [2]. The first family of identities is:

$$
\begin{aligned}
3^2 + 4^2 &= 5^2 & (k = 1)\\
10^2 + 11^2 + 12^2 &= 13^2 + 14^2 & (k = 2)\\
21^2 + 22^2 + 23^2 + 24^2 &= 25^2 + 26^2 + 27^2 & (k = 3)\\
&\vdots
\end{aligned}
$$

For each $k = 1, 2, \ldots$, Michael Boardman [1] identified a $(5k)$-piece symmetrical dissection for the k-th identity in the sequence. Boardman's dissection for $k = 1$, in Figure 6(a), illustrates his basic technique:

cutting the middle square into rectangles that are arranged around the smaller square(s) to produce the larger square(s). We have discovered $(4k)$-piece hinged dissections [15]; see Figure 6(b) for $k = 1$. Among all dissections whose cuts are parallel to the sides of the squares, ours use the fewest possible number of pieces.
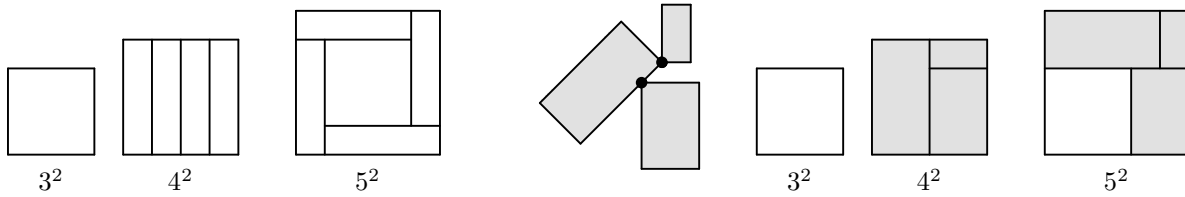

Figure 6: (a) Boardman's dissection for $3^2 + 4^2 = 5^2$ (b) My hinged dissection of squares for $3^2 + 4^2 = 5^2$

Applying Dostor's technique in a different way, we identified the sequence of identities below and also found hinged dissections that use $4k$ pieces, the fewest possible number of pieces [15].

$$
\begin{aligned}
1^2 + 2^2 + 2^2 &= 3^2 & (k = 1)\\
4^2 + 5^2 + 6^2 + 6^2 &= 7^2 + 8^2 & (k = 2)\\
9^2 + 10^2 + 11^2 + 12^2 + 12^2 &= 13^2 + 14^2 + 15^2 & (k = 3)
\end{aligned}
$$
$$\vdots$$

We can apply the same sort of approach for cubes, where we have identified the following sequence of identities. For each $k = 1, 2, \ldots$, we have discovered a $(9k)$-piece symmetrical dissection for the k-th identity [13]. It is analogous to Boardman's technique for squares, in that it cuts one cube into rectangular blocks that we then used to surround the smaller cubes. Robert Reid (in [5]) found an 8-piece dissection for the first identity, and we have shown how to generalize his approach to give $(8k)$-piece dissections for all of the identities [13]. Among all dissections whose cuts are parallel to the sides of the cubes, these dissections use the fewest possible number of pieces.

$$
\begin{aligned}
1^3 + 1^3 + 5^3 + 6^3 &= 7^3 & (k = 1)\\
1^3 + 1^3 + 2^3 + 2^3 + 16^3 + 17^3 + 18^3 &= 19^3 + 20^3 & (k = 2)\\
1^3 + 1^3 + 2^3 + 2^3 + 3^3 + 3^3 + 33^3 + 34^3 + 35^3 + 36^3 &= 37^3 + 38^3 + 39^3 & (k = 3)
\end{aligned}
$$
$$\vdots$$

In our parade, animations that emphasize the symmetry and similarity are given for the first two identities in each of the above three families. Especially captivating are the animations that emphasize the symmetry by "dealing off the bottom of the deck."

5. Folding multi-level regular polygons

We can use long hinges, such as the hinge that connects the lid to the body of a grand piano, to achieve a folding motion. Such hinges allow us to fold regular polygons of a certain number of levels to regular polygons of a perhaps different number of levels [14, 16].

In Figure 7 we see how to dissect and fold our 4-level equilateral triangle to a 3-level equilateral triangle [16]. A dotted line denotes a folding hinge between two pieces that are adjacent on the same level. When a
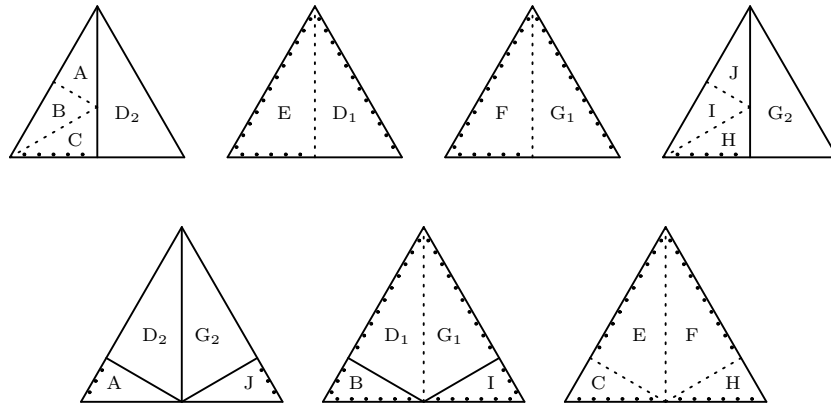
Figure 7: Folding dissection of a 4-level triangle to a 3-level triangle (top levels on left, bottom levels on right)

piece on one level is hinged to a piece on a level either immediately above or below it, we denote the hinge by a row of dots next to the shared edge on each level. In Figure 7, pieces D and G are two levels thick, and the remaining eight pieces are each one level thick. There is a cyclic hinging amongst pieces D, E, F, and G.

We have discovered several infinite families of folding dissections. For $k = 1, 2, \ldots$, we know how to fold a 1-level equilateral triangle to a k^2-level equilateral triangle, and also a 2-level equilateral triangle to a $(2k^2 + 2k + 2)$-level equilateral triangle [14].

For $k = 1, 2, \ldots$, we can also fold a 2-level $\{(4k+2)/(2k-1)\}$ star to a 4-level $\{(2k+1)/k\}$ star using $6k + 3$ pieces [14]. For $k = 2$ this gives the 15-piece folding dissection of a 2-level $\{10/3\}$ star to a 4-level $\{5/2\}$ star in Figure 8. Pieces B, E, H, K, and N are two levels thick, and the rest are one level thick.
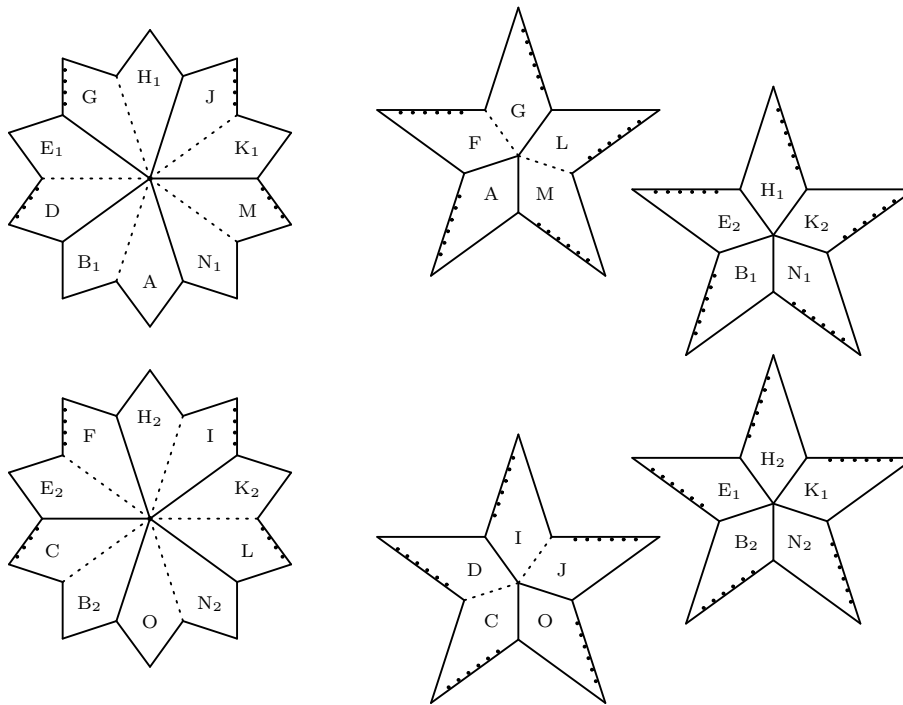

Figure 8: Folding a 2-level $\{10/3\}$ star to a 4-level $\{5/2\}$ star (piece A on the top, piece O on the bottom)

Our dissections and animations include: a 1-level square to an 8-level square, a 1-level regular hexagon to a 3-level regular hexagon, a 4-level regular hexagon to a 3-level regular hexagon, a 1-level {6/2} star to a 3-level {6/2} star, a 1-level equilateral triangle to a 4-level equilateral triangle, a 1-level equilateral triangle to a 9-level equilateral triangle, a 2-level equilateral triangle to a 14-level equilateral triangle, a 4-level equilateral triangle to a 3-level equilateral triangle, and a 2-level {10/3} star to a 4-level {5/2} star.

6. Folding multi-level polyominoes

A *polyomino* is a geometric figure consisting of congruent squares that are glued together edge-to-edge. We have discovered several infinite sets of polyominoes where each such polyomino can fold from two levels to one level of the same (*i.e.* similar) polyomino [17]. The most fundamental of these sets are the *well-formed* polyominoes, in which two HV-squares (squares that are adjacent both horizontally and vertically to other squares) are never adjacent to each other. See Figure 9 for two examples of well-formed polyominoes.
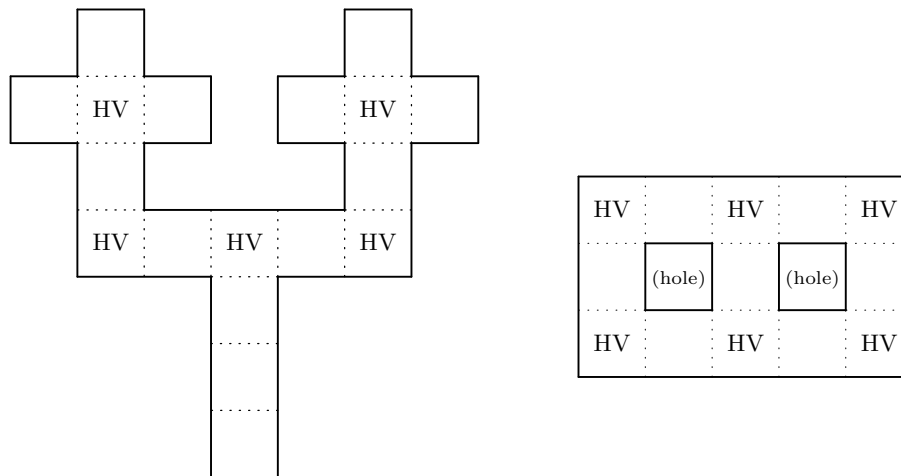

Figure 9: Two examples of well-formed polyominoes, with HV-squares marked

We are able to fold any 2-level well-formed polyomino to a 1-level polyomino of similar shape. Our animations include examples of 1-level to 2-level folding dissections of a Latin Cross, a Cross of Jerusalem, a Greek Cross, a Cross of Lorraine, an F-pentomino, and a quasi-well-formed polyomino. We also have an animation of a 1-level Greek Cross to 9 levels. In the video of our wooden model of the Cross of Jerusalem, each level of the 2-level version is a different color, which then folds into a 1-level version in which the pieces alternate in colors. Such a property holds for every well-formed polyomino.

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