Celebrating Mathematics in Stone and Bronze: *Umbilic Torus NC vs SC*

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Introduction

We celebrate mathematics with sculpture and sculpture with mathematics [3], [7], [6], [2]. What is unique about the mathematics that we want to celebrate with sculpture? Probably unseen infinite continua. For example, what is a celebration without π? Circles and π are ubiquitous in our sculpture, our umbilic tori, and elsewhere in mathematics.

Geometry and Arithmetic. An elegant theorem about π gives the Leibnitz alternating sum of the reciprocals of the odd integers as an Euler product over odd primes:

$$\frac{\pi}{4} = \prod_{2n+1\, prime} \Big(1 - \frac{(-1)^n}{2n+1}\Big)^{-1} .$$

The lefthand side is pure geometry and involves the irrational and transcendental real number π. The right-hand side is pure arithmetic and involves the infinitude of all prime numbers.

That geometry and arithmetic equate in this beautiful manner speaks in my mind as to what is mathematics. What we want to celebrate is compelling relationships and infinities.

To clarify these formulae, write out the first few terms of the product and sum,

$$\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdots\frac{2n+1}{2n+1-(-1)^n}\cdots = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\cdots+\frac{(-1)^N}{2N+1}+\cdots ,$$

where $2n+1$ is an odd prime and $2N+1$ is a positive odd integer. This Euler product = Leibnitz sum equality is quite an amazing arithmetic statement by itself. Caution: both the infinite product and the infinite sum are conditionally convergent, infinite arithmetic operations are not commutative! To further clarify the meaning of these formulae, recall that for an odd prime $p = 2n+1$ the function $\chi(p) = (-1)^{\frac{p-1}{2}}$ generates a multiplicative character and the equality between the product and series amounts to quadratic reciprocity. Back to the geometry side, my PSLQ algorithm discovered a rapidly convergent expression for π, evidently unknown to Euler:

$$\frac{\pi}{4} = \sum_{k\geq 0} \frac{1}{16^n}\Big(\frac{1}{8k+1} - \frac{1}{16k+8} - \frac{1}{32k+20} - \frac{1}{32k+24}\Big) = \frac{47}{60} + \frac{53}{26208} + \frac{829}{20106240} + \cdots .$$

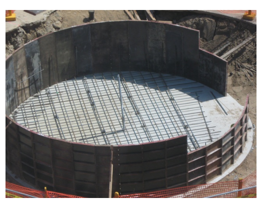
But that is another story and another sculpture (see [7] and the references therein).

Stone and bronze. Stone presents a medium to carve the negative to cast the positive in bronze. *Silicon bronze* is an alloy of copper with silicon. A typical recipe of this silicon bronze is composed of the 'molecule' $9438Cu + 430Si + 126Mn + 4Fe + Zn + Pb$. I think of the $430Si + 126Mn + 4Fe + Zn + Pb$ piece as being the 'stone' part. I pour these lava-like molten molecules into sandstone negative shapes which solidify into bronze positive shapes. The positive shape I designed is buried in the sandstone. After much shaking, pounding, and other violence the silicon bronze is broken out of the mold and can then be welded to other such parts. This sand casting is an industrial process distinct from the classical art process of lost wax bronze casting.

Plan of this paper. We briefly describe my early *Umbilic Torus NC* design and process. The *NC* refers to *numerical control*. The process differs considerably from the much larger, hole-filling *Umbilic Torus SC*. The much larger *Umbilic Torus SC* is also a type of *Umbilic Torus NC*. Mathematics tends to be scale invariant for the observer. Here we use the suffix *SC* for Simons Center for Geometry and Physics. This sculpture is a gift from the Simons Foundation to Stony Brook University. The project required that I orchestrate a large team of wonderfully talented friends and colleagues; I say *we* as a reference to that team. Here is a partial list: Cold Spring Granite quarry and mill in Minnesota, Berardi Stone Setting Inc of White Plains New York, the industrial sand-casting foundry Danko/Arlington Inc of Baltimore Maryland, the fine art New Arts Foundry Inc in Baltimore, the automation Eagle Engineering Corporation in Baltimore, my sculpture studio and Torus Assembly Building in Baltimore, the architects of Beyer, Blinder, Bell in New York, the structural engineers of Gilsanz, Murray, Steficik in New York, Campus Planning of Stony Brook University in Long Island New York, and the Simons Foundation in New York. As of our writing in May 2012, the *Umbilic Torus SC* is not yet completed. According to plan it will be installed by the time of Bridges 2012 in late July. We sketch events up to the final stages.

Filling a Hole

We dug a 36 ft. diameter and 8 ft. deep hole (first photo below) in the landscape at Stony Brook University. I fill up the hole with mathematical theorems expressed in a couple hundred tons of concrete, steel and lightning grounding cables (second and third photos below), sixty tons of granite (fourth photo below), and later, about fifteen tons of silicon bronze (not shown).

The two billion year-old granite on top of this concrete was quarried near Elizabethville in northern Minnesota near Lake Superior.





A base of five dozen tons of Lake Superior Green Granite surrounds the stainless steel column. The granite is cut into a deltoid (echoing the radial section of the *Umbilic Torus SC*) atop a 24 ft. circular disk with polished vertical sides for the inscribed text. I line the incisions for the characters with my usual pregnant cactus beetle (cochineal) perfect red, which lasts millenia. I designed the granite deltoid and disk base to be comfortable. I was not surprised that this base was put into use immediately by the Stony Brook students engaged in a humans-and-zombies game. Did you know these comfortable zombies are open source? Cf., [5].

But first of all, when confronted with such a big hole to fill, what was my first step? Build a big industrial grade gantry robot, of course.

Umbilic Torus NC

Over twenty-five years ago, I created *Umbilic Torus NC* in an elegant, antique verde, silicon bronze [3], [4].

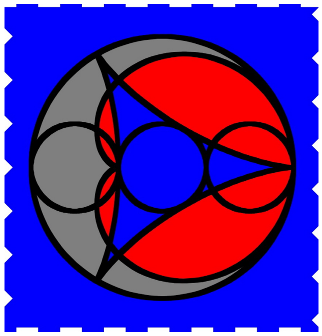


The two *Umbilic Tori NC* shown have antique verde patinas, with an engraved Peano-Hilbert Curve space-filling motif. These images and others like them have appeared over the years on covers of calculus books. The *NC* refers to numerical control, from a time when milling machines read their G-code instructions from paper tape. At that time the NC machines gave me a new tool to accurately carve the Hilbert Curve (a 2-adic Peano curve) as a surface-defining tool path [1].

An *Umbilic Torus* form with the non-triangular, three cusp deltoid radial cross-section interests me among all twisted toroids because of its explicit connection with the representation theory of the 2×2 invertible real matrices, i.e., $GL(2, R)$. These linear transformations act on binary (two variables x, y) quadratic forms $ax^2 + bxy + cy^2$ (three coefficients a, b, c). Such group actions yield ellipses, hyperbolas, and parabolas, e.g., simple forms like $x^2 + y^2$, $x^2 - y^2$, and x^2 respectively. Identifying a given quadratic in the form $ax^2 + bxy + cy^2 + dx + ey + f$ with one of the three is based on the quadratic discriminant $b^2 - 4ac$. Most students encounter these ideas in their calculus texts.

The group of linear transformations of two variables x, y, $GL(2, R)$, can also act on homogeneous binary (two variables x, y) cubic forms $ax^3 + bx^2y + cxy^2 + dy^3$ (with four coefficients a, b, c, d). A more interesting stratification of elliptic umbilics, hyperbolic umbilics, parabolic umbilics, and pure cubes emerges, corresponding to simple forms like $x^3 - xy^2$, $x^3 + xy^2$, x^2y, and x^3, respectively. The word umbilic is obscure but the anatomical reference is valid enough. For details see my paper [4] where I have written on the math of umbilic tori and referenced its history from -9000 until $+1990$. The algorithm for identifying a given cubic form $ax^3 + bx^2y + cxy^2 + dy^3 + \cdots$ with one of the four is based on the cubic discriminant $b^2c^2 - 4ac^3 - 4b^3d + 18abcd - 27a^2d^2$. Most students do not encounter such ideas in a calculus text, unless a picture of one of my umbilic tori appears on the cover.

A visualization of the above stratification emerges in the three sphere with deltoid cross-sections and cardioid cross-sections. The three cusps of the deltoid form a continuous curve running thrice the long way and once the short way around the torus. Points at infinity were chosen in this north-to-south pole stereographic projection of four space (a, b, c, d) into the three sphere $a^2 + b^2 + c^2 + d^2 = 1$, hence there is a pair of umbilic tori, each the inside out of the other.

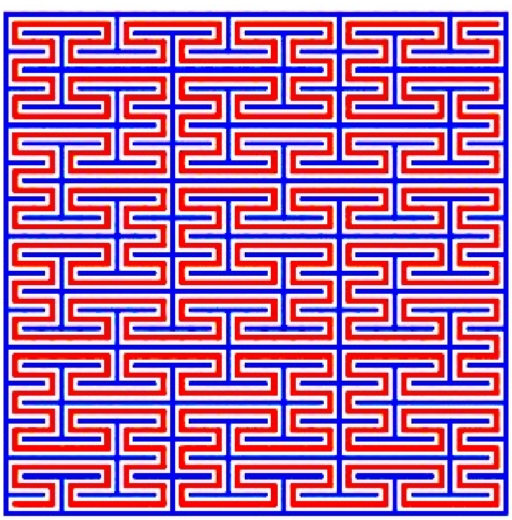
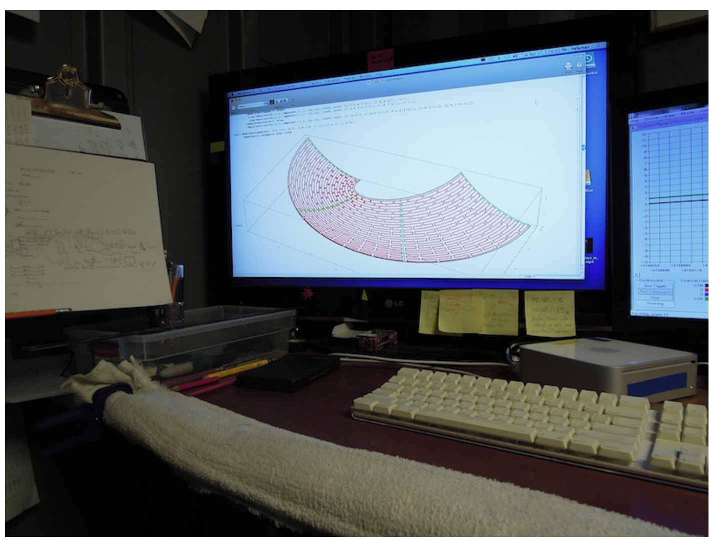




Here two umbilic tori roll through each other, radial deltoid touching sagittal cardioid and sagittal deltoid touching radial cardioid. The photos above show my sculptural pair as each the inside out of the other, the radial cross-section of one is the sagittal cross-section of the other, the curve of cusps of one has cohomology $(1, 3)$ for the one and $(3, 1)$ for the other. These linking tori roll through each other giving a hands-on, dynamic differential geometric view of a three sphere in four dimensions.

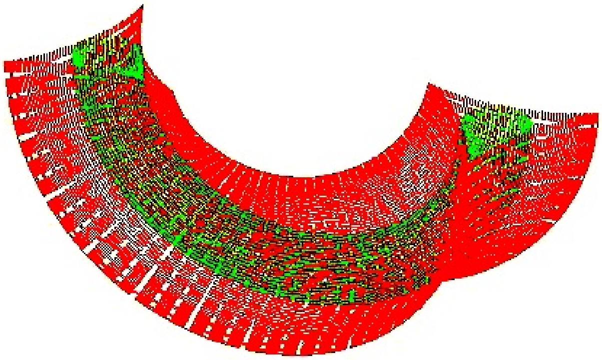
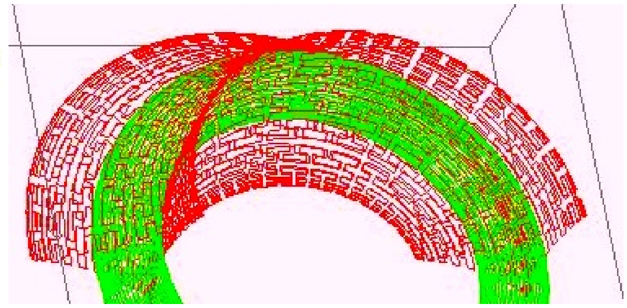
With respect to physically creating these two sculptures, the three main differences between *Umbilic Torus NC* and *Umbilic Torus SC* are: for *NC* I used a standard three axis numerically controlled milling machine carving styrofoam, for *SC* we built and I programmed a special purpose five axis numerically controlled sandstone carving machine, *NC* relied on the lost wax casting process, *SC* used direct-carved sandstone molds and sand casting, *NC* followed a 2-adic surface filling Peano-Hilbert tool path, *SC* followed a 3-adic surface filling Peano tool path to define the twisted torus surface.

Umbilic Torus SC

I carved the $(3, 1)$ twisted torus with radial cross-section deltoid (hypocycloid with three cusps) and sagittal cross-section cardioid (epicycloid with one cusp). I then articulated the surface with a Peano curve-based tool path [8], for more than one reason.

In the left photo above the 3-adic Peano Curve is red and the 3-adic Peano Maze is blue. In the right photo is my Mathematica^{TM} screen and its Mac computer, further right is part of the oscilloscope screen for the robot and controllers with its own Mac computer.

This two-dimensional curve needs to be wrapped around a torus with radial deltoid cross-sections, thrice the long and once the short way round. These two photos show the ghost trajectory of a space-filling 3-adic Peano curve tool path as it wraps around and *defines* the umbilic torus. The complete curve (in the limit) defines the surface everywhere. Starting at any point, following the curve, you will return to where you started. As you do so you will follow the 3-adic path for the diamond cutting tool just as I programmed it.

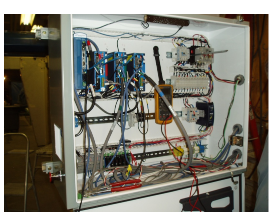
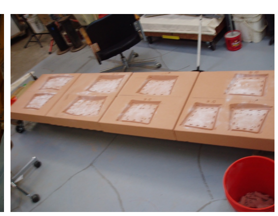
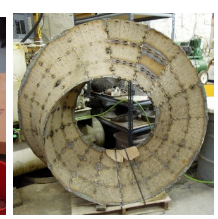
Sculpture as a fibre bundle. To create this sculpture, I wrote an intricate Mathematica$^{\text{TM}}$ program to generate G-code for the controllers of my giant gantry robot. To even begin to write this program I had to face the fact that the geometry of this sculpture consists of more than a twisted toroidal surface in space. I view it as a fibre bundle

$$T(3) \bowtie O(3) \times \mathbb{U} \underset{p}{\overset{s}{\leftrightarrows}} \mathbb{U}$$

where $\mathbb{U}$ represents the open umbilic torus surface sans cusps with the $T(3) \bowtie O(3)$ group as fiber, $T(3)$ is the XYZ translation group and $O(3)$ is the ABW group of Euler angle roll, pitch, yaw.

You expect to see the image $\mathbb{U}$ of the projection p going left to right as a sculpted surface. You don't see or feel the image of the section map s going right to left unless you are doing the actual carving. The section map is a critical part of my creative process, it specifies the continuous orientation of the foot-long diamond cutting tool in the grip of the six-axis robot and its XYZBW controllers.

Building a big gantry robot. The framework for this gantry robot has a footprint of sixteen feet by twenty feet. Two years ago it was lying in a thirty-ton heap of disorganized steel parts on the floor of our Baltimore studio. In the three photos below we see the gantry robot in its early stages of development,

with new XYZ and BW (pitch and yaw) motors and controllers. Building a gantry robot is a huge engineering task in itself. The last photo shows a quarter size proof of principle in bronze. The three XYZ axes specify the tool tip position, but not orientation. To orient the tool so it is perpendicular to the surface of the torus, the ABW axes are required in some form. Since the tool itself is rotating counterclockwise at 1740 rpm, the roll axis A is accounted for, and we added BW axis motors, a pitch (B) motor and a yaw (W) motor and their controllers. I designed this gantry robot to carve 144 one ton blocks of sandstone.



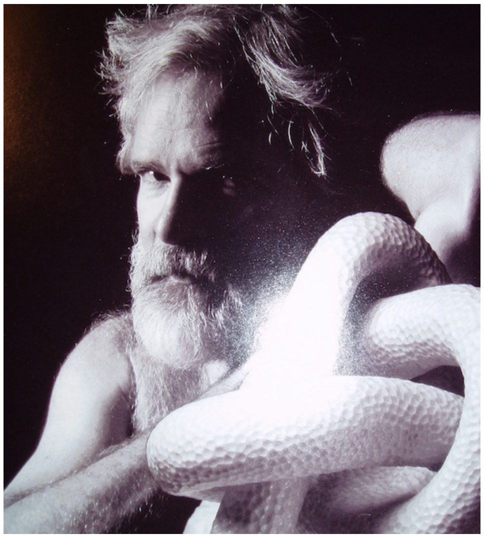
The robot is on the left, middle is the orientable diamond coated cutting tool, right is pouring the bronze into the sandstone molds. This gantry robot has a name, $\Sigma\tau\alpha\mu\alpha\tau\iota\alpha$ = 'fallen angel', it fits.


The next two photos show a few of the 144 curvilinear parts of the torus.


We carved each of the 144 parts in reverse with a set of adjacency labels so that they could be welded together without further instructions or pictures. We sand cast all 144 parts and have welded the top half of 72 parts and the bottom half of 72 parts. The final four photos give an idea of where things are in the Torus Assembly Building.

The top and bottom halves will be assembled outside the Torus Assembly Building in Baltimore before dis-assembly and shipping the two halves for re-assembly in Stony Brook • • • .

Authors

Authors. Claire and Helaman Ferguson received the *2002 Joint Policy Board for Mathematics Communications Award*.

References

[1] Jordan J. Cox, Yasuko Takezaki, Helaman R. P. Ferguson, Kent P. Kohkonen, and Eric L. Mulkay. Space-filling curves in tool path applications. *Computer-Aided Design (CAD)*, Special Issue: NC Machining and Cutter Path Generation:1–19, February 1994.

[2] Florence Fasenelli. *The Princeton Companion to Mathematics*, volume VII.14, chapter Mathematics and Art: The United States and Helaman Ferguson, page 952. Princeton University Press, Princeton and Oxford, 2008.

[3] Claire Eising Pratt Ferguson. *HELAMAN FERGUSON: Mathematics in Stone and Bronze*. Meridian Creative Group, Erie, Pennsylvania, 1994.

[4] Helaman R. P. Ferguson. Two theorems, two sculptures, two posters. *American Mathematical Monthly*, 97(7):589–610, August-September 1990.

[5] Jonathan Ferguson. Ferguson explains how zombies are open-source. YouTube video available from `http://www.youtube.com/watch?v=-j3M6Yk_GrQ` and `http://www.youtube.com/watch?v=mys6REEiChU`, April 2010.

[6] Helaman and Claire Ferguson. *The Best Writing on Mathematics 2011*, chapter Celebrating Mathematics in Stone and Bronze, pages 150–168. Princeton University Press, Princeton and Oxford, 2012.

[7] Helaman and Claire Ferguson. Celebrating mathematics in stone and bronze. *NOTICES of the American Mathematical Society*, 57 (7):cover, 840–850, August 2010.

[8] Giuseppe Peano. Sur une courbe, qui remplit toute une aire plane. *Mathematische Annalen*, 36 (1):157–160, 1890.

A complete bibliography can be found at the website http://www.helasculpt.com