Simple Rules for Incorporating Design Art into Penrose and Fractal Tiles

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Abstract

Incorporating designs into the tiles that form tessellations presents an interesting challenge for artists. Creating a viable M.C. Escher-like image that works esthetically as well as functionally requires resolving incongruencies at a tile's edge while constrained by its shape. Escher was the most well known practitioner in this style of mathematical visualization, but there are significant mathematical objects to which he never applied his artistry including Penrose Tilings and fractals. In this paper, we show that the rules of creating a traditional tile extend to these objects as well. To illustrate the versatility of tiling art, images were created with multiple figures and negative space leading to patterns distinct from the work of others. [1]

1 Introduction

M.C. Escher was the most prominent artist working with tessellations and space filling. Forty years after his death, his creations are still foremost in people's minds in the field of tiling art. One of the reasons Escher continues to hold such a monopoly in this specialty are the unique challenges that come with creating Escher type designs inside a tessellation[15]. When an image is drawn into a tile and extends to the tile's edge, it introduces incongruencies which are resolved by continuously aligning and refining the image. This is particularly true when the image consists of the lizards, fish, angels, etc. which populated Escher's tilings because they do not have the 4-fold rotational symmetry that would make it possible to arbitrarily rotate the image $\pm$ 90, 180 degrees and have all the pieces fit[9]. Instead, they require creating a complement for every edge even when exploiting their bilateral symmetry such as in *Angels and Devils*[5]. This is true for any tile that incorporates such an image.

A collection of papers in honor of Escher, *M.C. Escher's Legacy: A Centennial Celebration* contains a comprehensive study of his work [16]. Of the articles emphasizing art in this collection, most authors produced tile images that continue the practice of having one dominant figure in a tile that is completely filled. Our paper shows that there are greater possibilities beyond this.

The rules to creating tiling art are straightforward, and we describe the process and apply it to the mathematical geometries that are not found in Escher's otherwise vast body of work. In particular, we create tiling art based on constructions that have gained prominence since Escher's time: Penrose Tilings and fractals. In the case of the latter, we also put a tessellation inside a fractal tile to allow for growth in various directions. We also deviate from Escher and others by having negative spaces between the figures and show the advantages of negative space.

As with other tessellations, current Penrose Tiling art tends to use only one or two dominant figures completely filling the tiles[14, 7, 2]. And pure fractal art, with a few exceptions[6], emphasizes the final mostly abstract image[3] rather than creating artistic designs inside a tile. This represents a missed opportunity as the geometry of tessellations and fractals can reveal other appealing patterns of density and void as seen in the images which follow.

[1] Another version of this paper was first published on the pre-print site **arXiv** [11].

2 Creating a Tessellation

To illustrate the process and challenges of making tiles that do not have 4-fold rotational symmetry, we begin with the simplest tessellation made of squares. Figure 1 shows a square tile in the *p1* wallpaper group with 4-fold rotational symmetry on the rhombi of its outer edges, and any $\pm$ 90, 180 degree rotation will position the rhombi the same as before. Any side *A* can be connected to another side *A* and the resulting tessellation after various rotations is given in Figure 2.

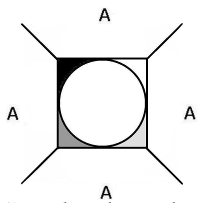

Figure 1 : *Simple tile with 4-fold rotational symmetry on outer rhombi*

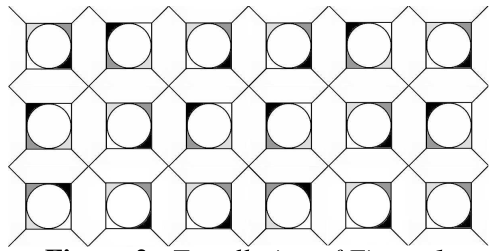

Figure 2 : *Tessellation of Figure 1*

In contrast, when the design is of a person as in *Leonardo da Vinci*'s *The Vitruvian Man* shown in Figure 3[13], it may not be possible for any side to connect to another. If the parts of his body extend to the boundaries of a tile, then there are four unique sides requiring that each side be complemented by another and the resulting tile is in the *p1* wallpaper group. To create a tessellation from Figure 3, we created a square, labeled its sides, and assembled parts of the *The Vitruvian Man* inside so that there was alignment of body parts across the edges. As seen in Figure 4, side *A* connects to side *A'* and *B* to *B'*. The arms and legs have also been color-coded to show how they will connect past the edge of the tile. The resulting tessellation is given in Figure 5.

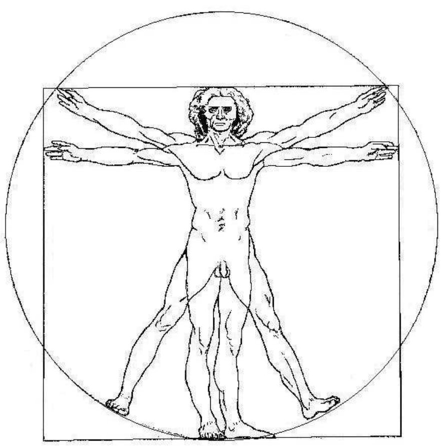

Figure 3: *Leonardo da Vinci's* The Vitruvian Man

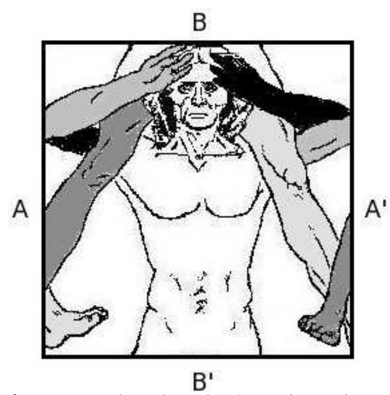

Figure 4 : *A single tile of* The Vitruvian Man *for tessellation*

3 Beyond Escher

Once the simple rules of making a tessellation are understood, other possibilities exist in how tiles are connected and the types of images they contain.

3.1 Other Images and Connecting Rules

Instead of limiting a tile to one prominent entity as typically seen in most work emulating Escher, Figure 6 contains multiple subjects and its rectangular tile, also in the *p1* wallpaper group, leads to the tessellation of

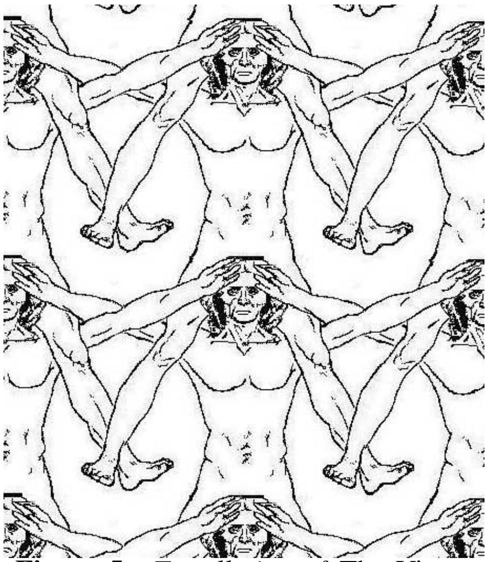

Figure 5 : *Tessellation of* The Vitruvian Man

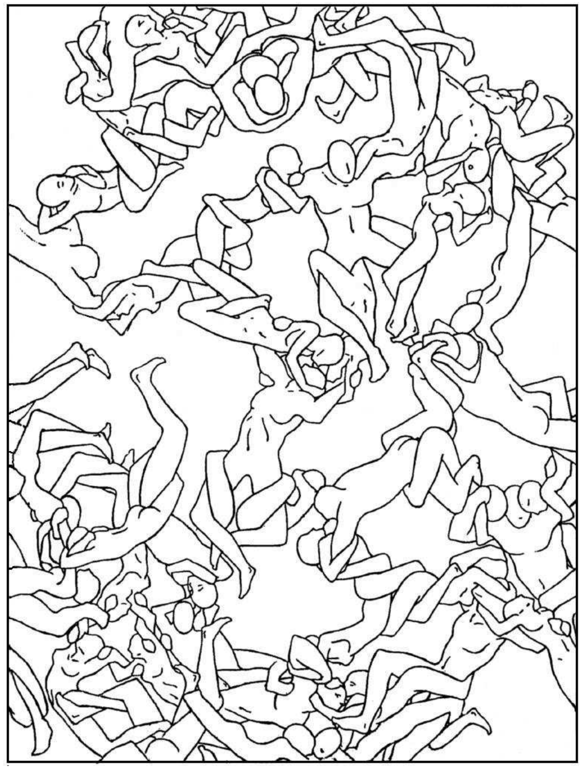

Figure 6 : *A tile made up of multiple figures*

Figure 7. The painted version of this tessellation, which can be seen at:

`http://www.greybodies.com/web3.html`

shows the value of allowing for negative space. The people intertwine and overlap, unlike in most completely space filled tiles where this is much harder to do leading subjects to be interlocked. Also, the connections between people are clearly seen when contrasted with a blank background. Escher himself seems to have been aware of these esthetics as shown by the woodcut, *Snakes* [5]. This is one of his few pieces where the subjects overlap and use of negative space greatly enhances the image's coherency.

The next two figures show other matching rules that square tiles can have. As seen in Figure 10 formed by the tiles of Figure 8 which fall in the p4 wallpaper group, by making the complementary sides adjacent, the final image forms an interesting pattern of swirling figures rather than a simple shifting of the tile vertically or horizontally.

In Figure 9, the same adjacent complements exist, but with each side complementing two sides instead of one. This leads to the image seen in Figure 11 which has the same turning of the figures, but also allows for non periodic arrangements. Once one row of tiles is set, the next row has two possible arrangements depending on how the first tile of the new row is placed. This is true for every subsequent row.

3.2 Penrose Tilings

Though Escher was friends with Roger Penrose, he died before applying his space filling work to Penrose's aperiodic tilings[4, 10]. These types of tilings are even more challenging to incorporate designs because they are made of two tiles which, depending on the arrangement, may not form a repeating pattern. Fortunately, the more interesting and non-repeating (hence aperiodic) arrangements are created by following side matching rules in the same manner as discussed before[12]. The Kite and Dart tiles, given in Figure 12, show how every side is connected to create the aperiodic tiling of 13. Using these same rules, the artistic tiles in Figure 16 lead to 18.

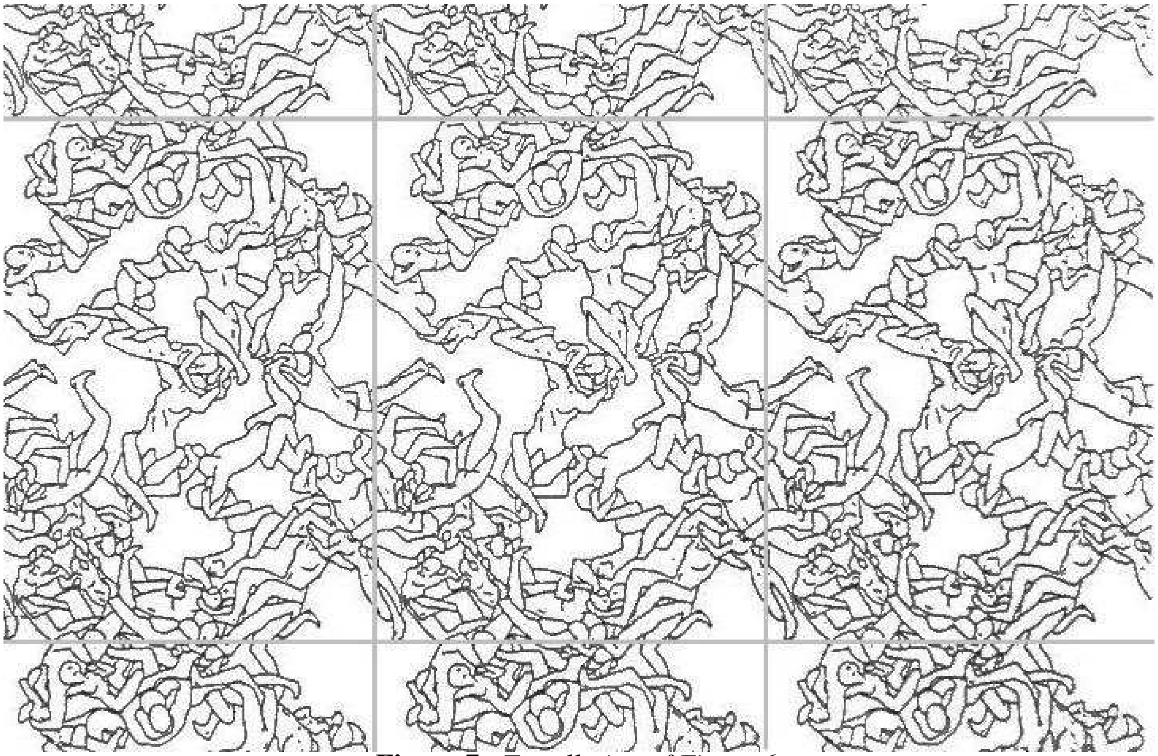

Figure 7 : *Tessellation of Figure 6*

The other Penrose Tiling made of fat and thin rhombi has similar rules for creating an aperiodic pattern. Figure 14 shows how the sides match to create 15. The individual tiles containing a design and the resulting image are given in Figures 17 and 19 respectively.

3.3 Fractal Tilings

For fractals, Escher's *Circle Limit* series invokes the concept but it can also be interpreted as a tessellation of the hyperbolic plane[1]. To make a fractal tiling that purely manifests the notion of self similarity, many of the previous techniques are applied again with a few modifications. The types of fractals which emerge are in the category of *Fractal Trees*[8]. Starting with a single tile, a matching rule is created for how other tiles will be connected to its sides. An additional constraint is that the added tiles are decreased in size by an amount that allows for growth of the fractal without the branches intersecting. Figure 20 shows one rectangular tile where subsequent additions decrease in size by 1/2. So the next tile is half of the original,

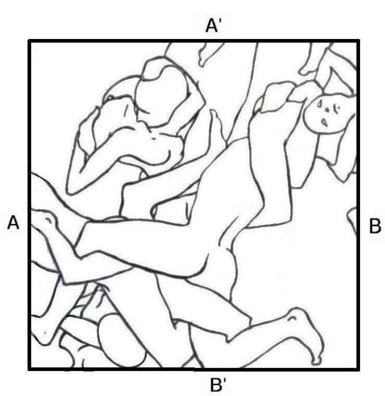

Figure 8 : *Tile having adjacent complementary sides*

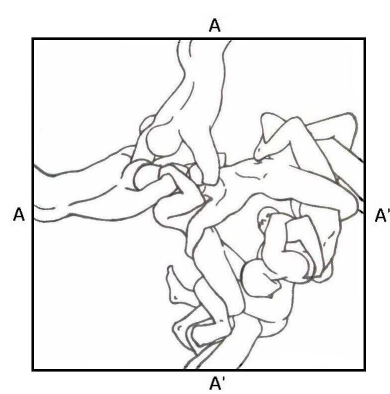

Figure 9 : *Tile having two adjacent complementary sides*

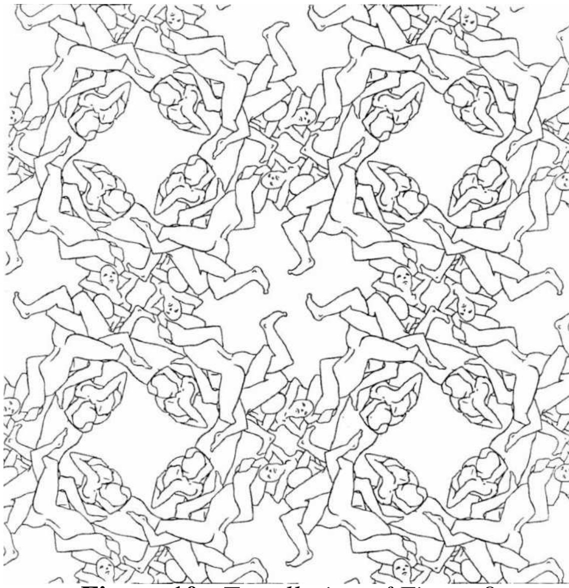

Figure 10 : *Tessellation of Figure 8*

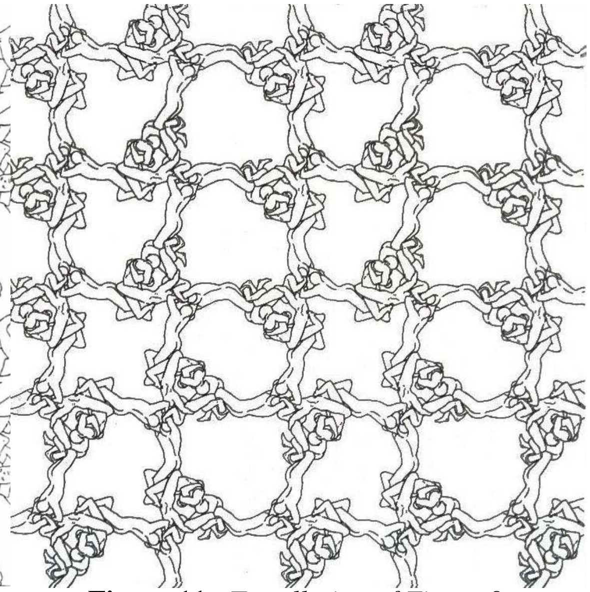

Figure 11 : *Tessellation of Figure 9*

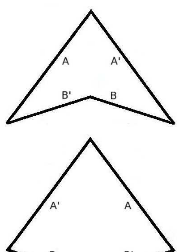

Figure 12 : *Kite and Dart Penrose Tilings*

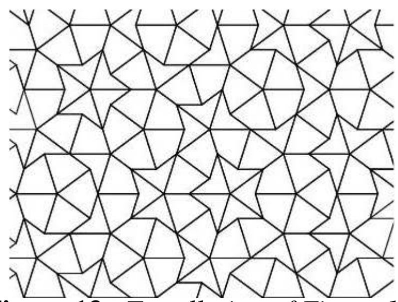

Figure 13 : *Tessellation of Figure 12*

and its side *A* will connect with *A'* and *A"*. To improve the esthetics, the two connecting sides are made to alternate such that side *A"* connects to the next tile after it is rotated 90 degrees clockwise while side *A'* connects to the mirror image of this tile and rotated 90 degrees counter-clockwise. Figure 21 shows the result.

For the next fractal tile made of an isosceles triangle, the same mirroring connection on side *A'* is made. In addition, the principals of tessellation are used so that the fractal can progress in other directions. As seen in Figure 22, the area around the center of the triangle is a region of symmetry for the design which is repeated every 120 degrees. This is delineated by 3 sub-triangles which form the tessellation inside the isosceles. From here, the lower triangle can be swapped with either of the other two leading to the fractal of Figure 23 which has branches in the downward direction. In this fractal, the decrease of each tile was only one third rather than one half which illustrates the problem of branches colliding. Despite this, the final

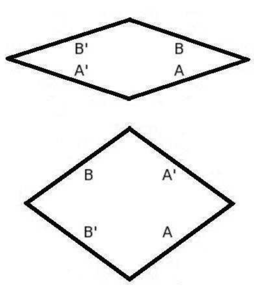

Figure 14 : *Rhombus Penrose Tilings*

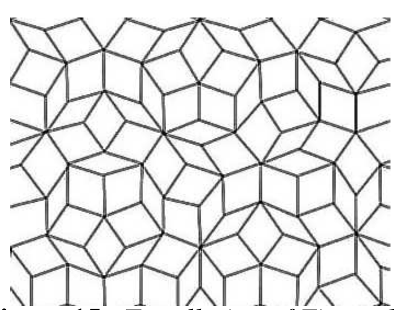

Figure 15 : *Tessellation of Figure 14*

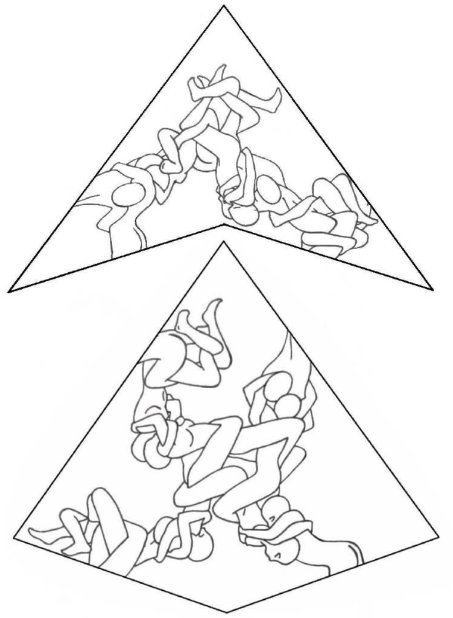

Figure 16 : *Decorated kite and dart Penrose Tiles*

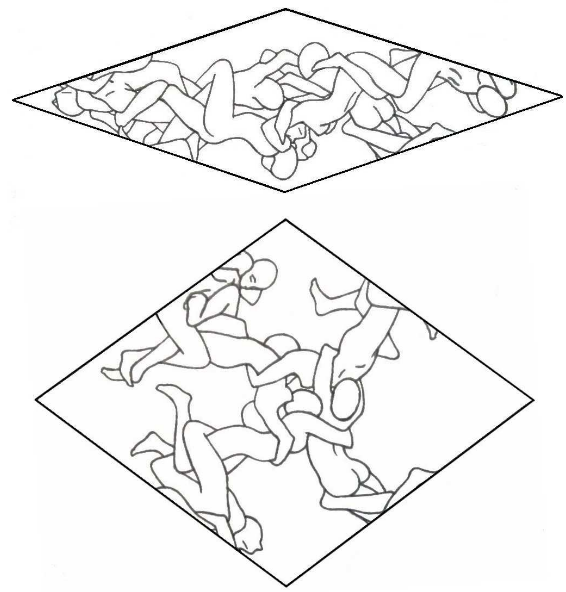

Figure 17 : *Decorated rhombus Penrose Tiles*

result is very appealing.

4 Concluding Remarks

Escher's work greatly expanded the possibilities of geometrical art by adding his wondrous talents of design to the realm of mathematics. But non-mathematician artists tended not to follow his example, and so a wealth of geometric shapes only exists as blank tiles waiting to be filled. By describing the process of incorporating tessellations and fractals into art, we hope to show that the challenges are artistic rather than mathematical.

In the same way that mathematical analysis can give a deeper appreciation of art, Escher used art as a means of gaining insight into the mathematics of geometry. Continuing this tradition, our paper demonstrates that the connections contained within tessellations and fractals are best seen when combined with tiling art. With the limitless possibilities as to what can be put inside a tile, artists are well suited to find the undiscovered pattern contained in each.

References

[1] B. M. Adcock, K. C. Jones, C. A. Reiter, and L. M. Vislocky. Iterated function systems with symmetry in the hyperbolic plane. *Comp.& Graph.*, pages 791–796, 2000.

[2] D. Bailey. David bailey's world of escher-like tessellations. *http://www.tess-elation.co.uk/penrose-tilings/thin-and-thick-rhombi*, 2009.

[3] R. Bandt and P. Gummelt. Fractal penrose tilings i. construction and matching rules. *Aequ. math.*, 53:295–307, 1997.

[4] R. A. Dunlap. Fivefold symmetry in the graphic art of m.c. escher. *Fivefold Symmetry*, 1994.

[5] B. Ernst. *The Magic Mirror of M. C. Escher*. 1995.

[6] R. Fathauer. Tree of knowledge. *http://mathartfun.com/shopsite_sc/store/html/Art/Tree0Knowledge.html*, 2002.

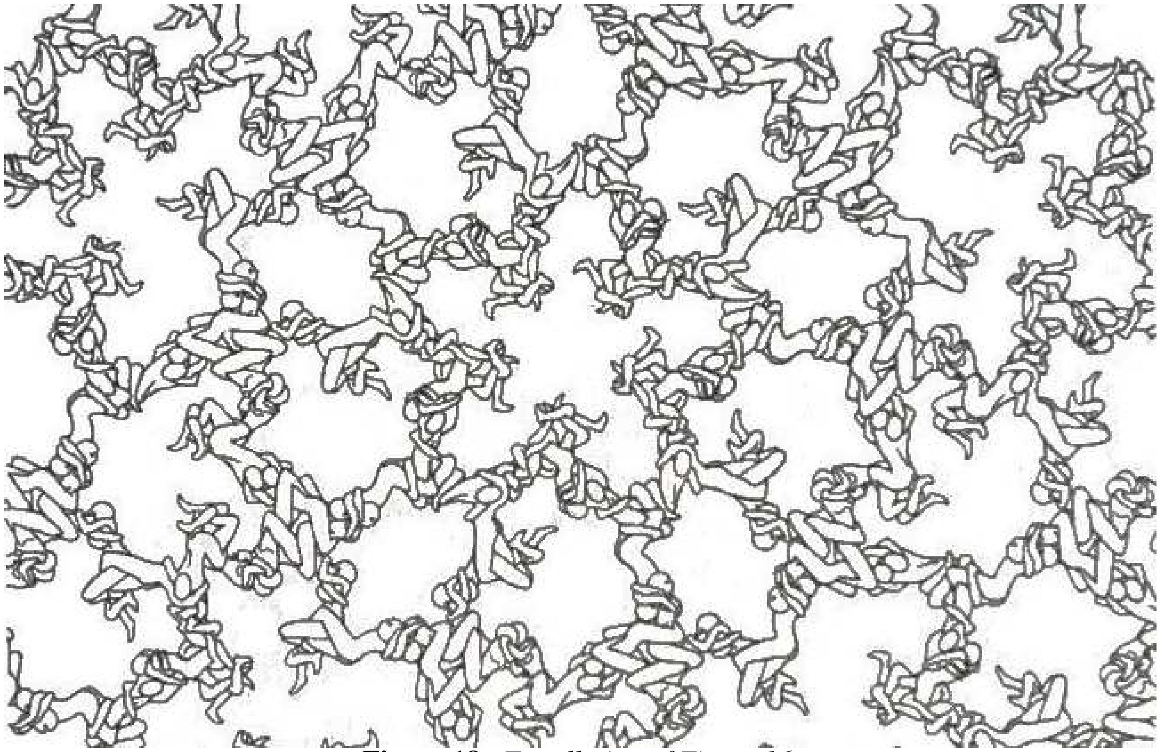

Figure 18 : *Tessellation of Figure 16*

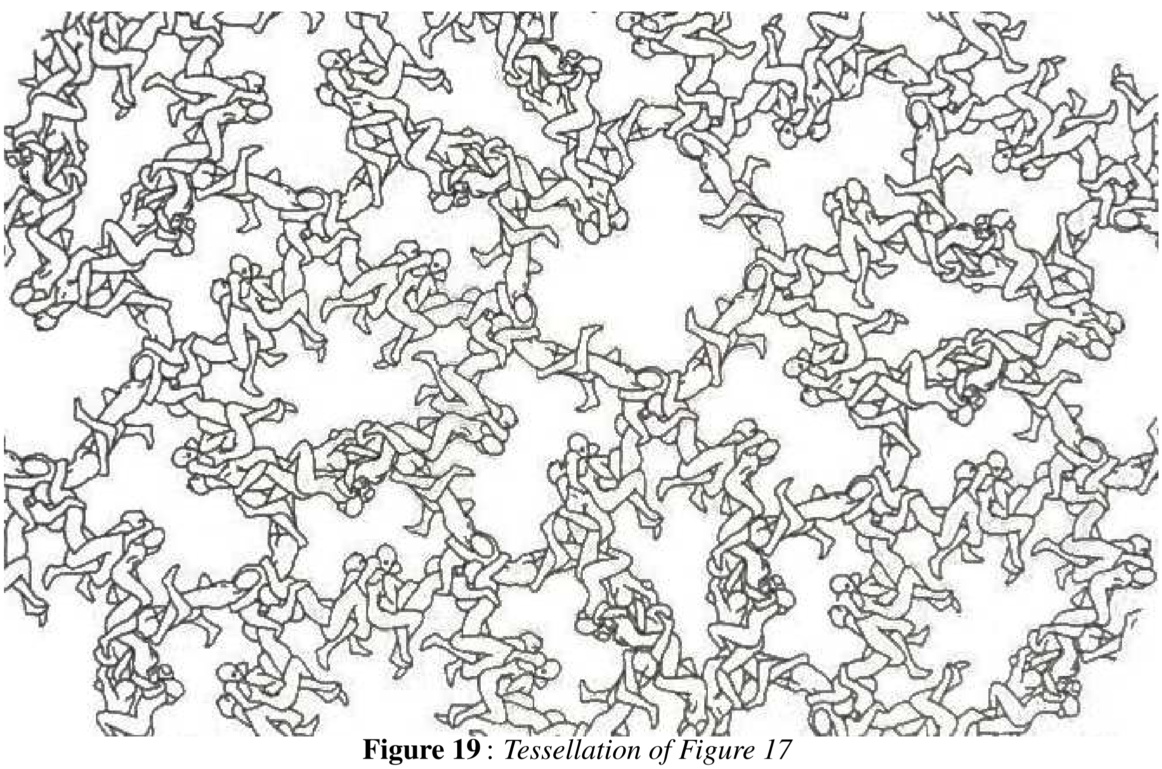

Figure 19 : *Tessellation of Figure 17*

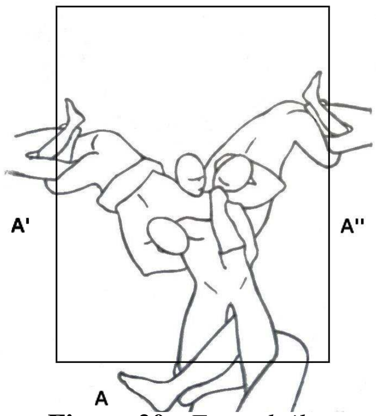

Figure 20 : *Fractal tile*

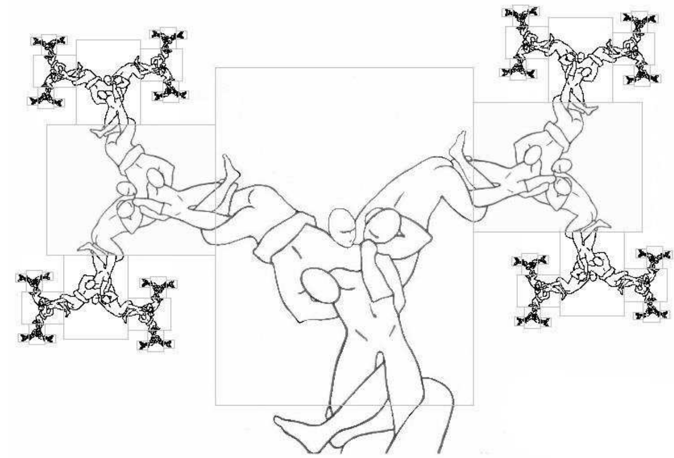

Figure 21 : *Final image using the tile of Figure 20*

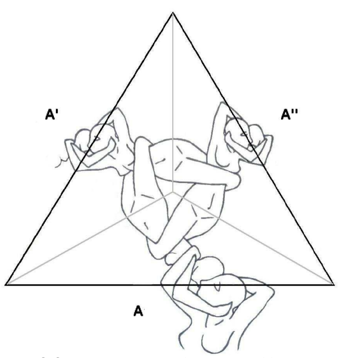

Figure 22 : *Fractal tessellation* *combination tile*

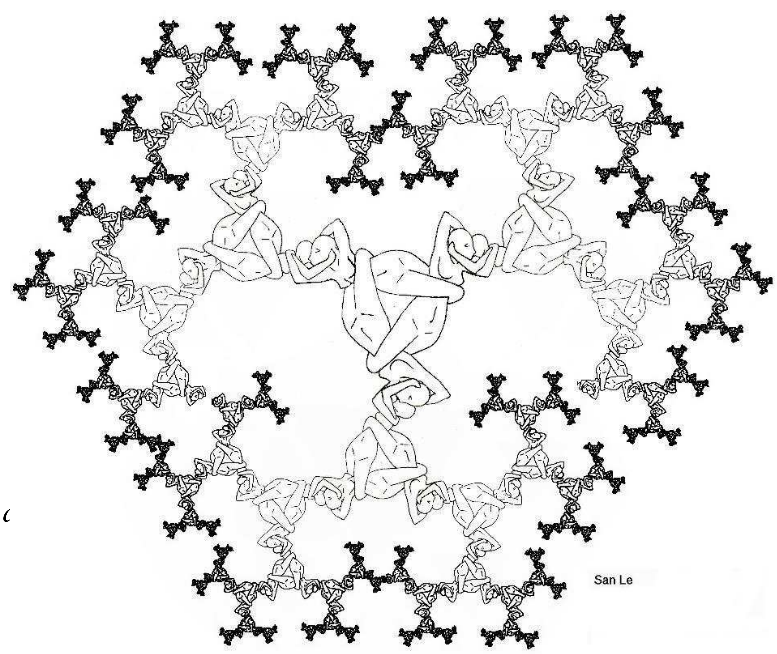

Figure 23 : *Final image using the tile of Figure 22*

[7] R. Fathauer. Extending escher's recognizable-motif tilngs to multiple-solution tilings and fractal tilings. *M.C. Escher's Legacy: A Centennial Celebration*, pages 154–165, 2005.

[8] R. Fathauer. *Fractal Trees*. 2011.

[9] M. James. *The Second Quiltmaker's Handbook: Creative Approaches to Contemporary Quilt Design*. 1996.

[10] C. S. Kaplan and D. H. Salesin. Dihedral escherization. *GI '04 Proc. Conf. Graphics Interface 2004*, pages 255–262, 2004.

[11] S. Le. The art of space filling in penrose tilings and fractals. *http://arxiv.org/abs/1106.2750*, 2010.

[12] E. A. Lord. Quasicrystals and penrose patterns. *Current Sci.*, pages 313–319, 1991.

[13] N. Mirzoeff. *Bodyscape: Art, Modernity and the Ideal Figure*. 1995.

[14] J. Osborn. ozbird.net penrose gallery. *http://www.ozbird.net/PG.html*, 2001.

[15] P. Raedschelders. Tilings and other unusual escher-related prints. *M.C. Escher's Legacy: A Centennial Celebration*, pages 230–243, 2005.

[16] D Schattschneider and M. Emmer. *M.C. Escher's Legacy: A Centennial Celebration*. 2005.