Mathematics and the Ballet Barre

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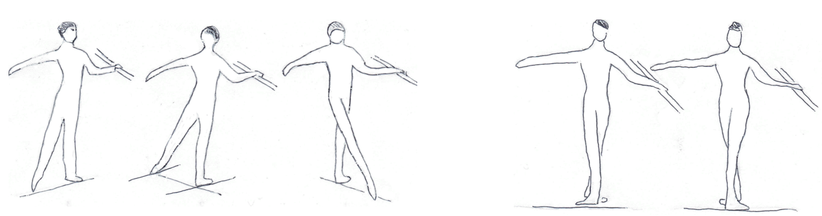
Abstract

The ballet barre is a standard set of exercises at the beginning of the traditional ballet class, and is usually composed of a small number of movements and positions arranged according to established constraints and format. In this paper we simplify and examine some mathematical properties of these exercises, focusing on the movement known as tendu, examine some connections to the Fibonacci numbers, and set up a finite-state automaton as a means of visualizing the exercises.

For over ten years the author has taken a ballet class from a teacher who, like many dance teachers, invents new variations on standard exercises every week. This teacher, Rebecca Blair, formerly of the Chicago City Ballet, has a penchant for permuting a small number of basic warm-up exercises so successfully that the author does not believe she has ever repeated herself in classes he has taken. In this paper we will examine permutations of a limited number of typical ballet barre warm-ups, using a non-deterministic finite-state automaton for visualization.

Let us first look at a simple puzzle based on the ballet movement known as "tendu" (Fig. 1). In a tendu, the dancer extends the leg either directly forward (*F*), to the side (*S*), or backwards (*B*), with a brushing and stretching motion, ending with the leg and ankle pointed and the toes on the floor. Suppose the dancer is standing in the usual opening position of the ballet barre warm-up, with dancer's left side at the barre. The barre itself is a horizontal pole set at approximately waist level, and the dancer's left hand rests on the barre for stability. The plane of the vertical torso is perpendicular to the barre and in what is known as "**fifth position**" the right heel is in front of and against the left toes, while the legs are "turned out," or rotated outwards the maximum possible for each dancer's body (*V*). In this position the right leg is free to gesture to the front or to the right side, and the left leg is free to gesture to the back (generally the left leg will not gesture to the left side, where the barre is!) In this puzzle we will assume the right leg gestures only to the front from the "closed" fifth position, and on the following movement closes again to fifth, repeating this sequence as many times as called for in the exercise.



S=Side *B=Back* *F=Front* *V* *W*

Figure 1: *Tendus and 5th position*.

Often the tendu is performed either quickly, let us say taking one musical count, or slowly, taking two counts. The reverse of the tendu, closing back to fifth position from the extended tendu postion, might also be executed in either one or two counts. A series of tendus, which alternate with closings back to fifth position, are often performed for exactly four or eight counts. In how many ways may an n count series of tendus be performed such that each movement takes one or two counts, and so that the phrase begins and ends in fifth position? For example, one such 8 count sequence is:

count 1: tendu; count 2: close; counts 3&4: tendu; count 5: close; count 6: tendu; counts 7&8: close

We might more succinctly notate this as 112112, since counts in odd positions must be tendus, and those in even positions must be closings to fifth. Our puzzle thus equivalently asks for the number of sequences of 1s and 2s, such that terms sum to n, and having an even number of summands, since an odd number of movements would leave the leg extended in tendu position. The table below shows the numbers of such sequences for n counts, from 0 to 8. E_n is the number of n count sequences with an even number of summands, O_n, the number with an odd number of summands, and T_n is the total number. We set E_0 to 1 since the empty sequence with 0 movements leaves the leg in fifth position. For n=5, for example, the even sequences are 1112, 1121, 1211, and 2111, and the odd sequences are 11111, 122, 212, and 221.

n	0	1	2	3	4	5	6	7	8
E_n	1	0	1	2	2	4	7	10	17
O_n	0	1	1	1	3	4	6	11	17
T_n	1	1	2	3	5	8	13	21	34

Figure 2. *Number of ways to perform n counts as sequence of 1s and 2s.*

T_n is equal to the $n+1^{st}$ Fibonacci number, the sequence usually numbered $F_1=1$, $F_2=1$,… $F_n= F_{n-1} + F_{n-2}$. This pattern was noticed earlier than Fibonacci by Hemachandra, and possibly earlier still by other Indian scholars in their examination of short (1 count) and long (2 counts) syllables in Sanskrit prosody and music [1]. Its occurrence is explained by the fact that if the first number in a sequence counted by T_n is 1, then the remainder of that sequence may be completed in T_{n-1} ways, while if the first number is 2 it may be completed in T_{n-2} ways, so $T_n = T_{n-1} + T_{n-2}$. The pattern repeats as follows:

(a) If $n = 3k$, then $E_n = O_n + 1 = (T_n+1)/2$.

(b) If $n = 3k+1$, then $E_n = O_n - 1 = (T_n-1)/2$.

(c) If $n = 3k+2$, then $E_n = O_n = T_n/2$.

The sequences E_n and O_n are sequences A094686 and A093040, respectively, of the Online Encyclopedia of Integer Sequences [2]. Not all sequences counted by E_n are equally useful as ballet barre warmups. For example, 211211, which repeats 211 twice, is easier to perform than 211121, even though the first 2 in 211211 is performed as a slow tendu and the second 2 is performed as a closing to fifth. One reason for this is that 8-beat music used for classes often accents beats 1,3,5, and 7, so the impulse for the 2s in 211211 would fall on downbeats, whereas the first 2 in 211121 would fall on a downbeat and the second on an upbeat; many dancers would find 211211 a nice challenge, while 211121 might produce a certain amount of annoyance, especially at the start of class!

Warm up. The ballet warm-up is almost always done standing next to a barre, Each exercise is typically done with the left hand and side adjacent to the barre first, then repeated with the right hand and side adjacent to it. Each exercise may be composed of a variety of movements, including pliés (flexing the knees to lower and then raise the body), relevés (pressing upwards so the weight is on the balls of the feet), and a series of gestures with the leg not adjacent to the barre. The upper body, head, and arms also accompany gestures of the legs in a clearly defined manner. In this brief article we will completely ignore positions or movements of the upper torso and arms, as well as pliés and relevés, focusing entirely on one basic movement known as tendu, performed from one of three simple standing positions.

During tendu exercises, standing with the left side towards the barre, the right leg may progress to the rear position, with left foot in front and the left heel against the right toes (W), allowing the right leg to then gesture to the rear or right side, while the left leg may then gesture to the front. Tendus are also performed from **first position** (I), in which the legs are turned out, and the body is bilaterally symmetric, with the heels of both feet placed against each other. From this position the right leg may freely gesture front, right, or back, and the left leg may gesture front or back. In order to be consistent with dance usage, in the remainder of this paper, we will primarily consider movements that take ½ or 1 beat, rather than 1 and 2 beats; most dance teachers actually use half beat movements, so that what we call the rhythmic sequence 112112 might more often be performed ½½1½½1, and verbally announced as "&,1,--2,&,3,--4," where --2" and --4" would be spoken slowly so as to take the same amount of time as "&,1." The "&" is used to signify a half-beat count. This is standard usage in dance, and often differs from the way musicians count music.

Figure 3 shows a non-deterministic finite-state automaton $M = \{S, A, g, V, V\}$ with states S, input alphabet $A=\{½,1\}$, transition function $g : S \times A \rightarrow P(S)$, and initial and final state V, representing the series of tendus in a warm-up exercise. We will simplify the model by assuming that each gesture either takes ½ or 1 beat. Each arrow joining the states is double headed, since any gesture in one direction may be reversed with the same timing. Pauses are shown by an arrow starting and ending at a state, since these dance sequences often include such pauses, which we will assume also take a half or full beat. The states in S are given by the following labels inside the circles:

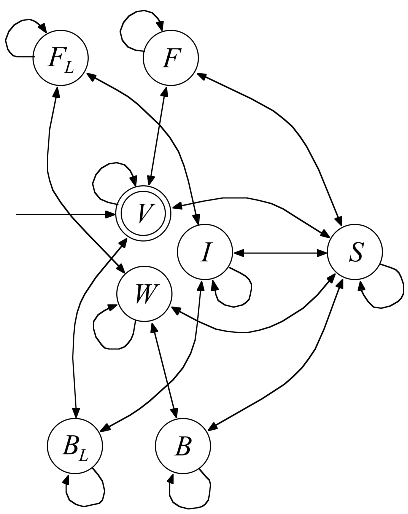


Figure 3: *Tendu finite automaton*.

V = starting and final state, in which dancer stands in ballet 5th position with right leg in front.
W = 5th position with right leg in back. I = Ballet 1st position.
$F = F_R$ = Right leg tendu to the front. F_L = Left leg tendu to the front.
$B = B_R$ = Right leg tendu to the back. B_L = Left leg tendu to the back.
S = Right leg tendu to the right side.

In this automata the starting and final state are both 5th position with the right leg in front, since such exercises usually do begin and end in this position. Note that F_L, a tendu to the front with the left leg, is accessible only from 5th position with the right leg in the back or from 1st position, and similarly B_L from 5th position front or first. A number of other constraints are often incorporated in these warm ups as well, for example, the number and pattern of gestures with the right leg to the front is repeated during the exercise to the back, and the same or a similar pattern may be done to the right side. In addition, the total number of beats is often a multiple of 8 beats. The movements we will consider here and shown by the automata are either tendus or their reversal, pauses, and the movements between F, S, and B, which are circling motions of the leg known as "rond de jambs." In this diagram and presentation there are three standing states in which the weight of the body is shared equally through both legs: V, W, and I. The other five states are those in which one of the legs is extended front, back or to the right side in tendu position.

Such exercises fulfill a number of functions: warming up the primary locomotor muscles in the lower body, slowly articulating joints in the legs and feet, practicing quick shifts of weight from one leg to the other, carefully rotating the legs through the hip joints, focusing concentration on the body at the start of class, as well as initiating the ritual of dance class itself.

Examples. A typical example of a recent tendu sequence from the class illustrates the types of actual patterns used. As is common for many such sequences, it consists of four sections of 8 beats each, in which the primary 8-beat pattern to the front is then repeated to the side, the back, and then the side again.

This pattern might be described verbally as "three tendus to the front on 5 counts, rond de jamb side, 6, close back, 7, tendu side, &, close front, 8. Repeat to the side, the back, and the side again."

For the purposes of this paper, we will signify dance sequences in which we are interested by listing the sequence of states, each with exponent either 1 or ½, for the number of beats of the movement terminating in that state. So the first eight beats of the exercise above could be notated $F^1V^1F^1V^1F^1S^1W^1S^{1/2}V^{1/2}$. The same pattern repeated to the side is notated $S^1W^1S^1V^1S^1B^1W^1B^{1/2}W^{1/2}$, Note that in this sequence the first tendu to the side is followed by fifth position with right leg back, W, then to the front, V; tendu patterns to the side often make use of this kind of alternation. To the back the pattern is $B^1W^1B^1W^1B^1S^1V^1S^{1/2}W^{1/2}$, followed by $S^1V^1S^1W^1S^1F^1V^1F^{1/2}V^{1/2}$ to the side again. Overall, the sequence could be listed

$$(F^1V^1F^1V^1F^1S^1W^1S^{1/2}V^{1/2})(S^1W^1S^1V^1S^1B^1W^1B^{1/2}W^{1/2})(B^1W^1B^1W^1B^1S^1V^1S^{1/2}W^{1/2})(S^1V^1S^1W^1S^1F^1V^1F^{1/2}V^{1/2})$$

If we treat the product as commutative and add exponents of factors with the same base, the product reduces to $(F^3S^{1.5}V^{2.5}W^1)(S^3B^{1.5}W^{2.5}V^1)(B^3S^{1.5}W^{2.5}V^1)(S^3F^{1.5}V^{2.5}W^1)$, or removing the parentheses and adding exponents of like bases, $F^{4.5}B^{4.5}S^9V^7W^7$.

The example above indicates a general overall pattern: $F^nB^nS^{2n}V^{16-2n}W^{16-2n}$, with total degree, or number of beats, 32. For n = 1 or 2 the patterns require a number of pauses, a fact that we will not prove here. Patterns for n = 3 are possible without pauses, but the alternation of half beat and full beat movements would be more complex than most standard warm ups.

A fairly simple tendu exercise for $n = 4$, in which all movements take one beat and so have exponent 1, is composed of 4 tendus, first to the front, then side, then back, and then side again; we have modified the use of exponents, so that here $(XY)^m$, with the exponent just outside the parentheses, will mean m copies of XY in which all timing exponents are 1: $(FV)^4(SVSW)^2(BW)^4(SWSV)^2$. Letting b stand for a one beat pause, another exercise is ($FVSWBWbb$)($SVFVSWbb$)($BWSVFVbb$)($SWBWSVFbb$). A pattern with gestures by both legs is $(FVSWBWF_LW)(SVFVSWF_LW)(BWSVFVB_LV)(SWBWSVB_LV)$.
A rhythmically challenging pattern, since the pauses occur at beat 4, then 3, then 2, then 1, is:

$$(F^{1/2}V^{1/2}S^{1/2}W^{1/2}B^{1/2}W^{1/2}b)(S^{1/2}V^{1/2}F^{1/2}V^{1/2}bS^{1/2}W^{1/2})(B^{1/2}W^{1/2}bS^{1/2}V^{1/2}F^{1/2}V^{1/2})(bS^{1/2}W^{1/2}B^{1/2}W^{1/2}S^{1/2}V^{1/2})$$

These exercise patterns have all appeared in Blair's classes.

It should be clear that even after reducing the possible movements by removing pliés, relevés, flexing of the feet, and other common ballet positions, there are a large number of possible warm up sequences of tendus, even when further constrained to a total of four 8-beat sections, and limited to movements of 1 or ½ beats. Many mathematical properties remain to be investigated. For example, teachers often use movements that take either ½, 1, or 2 beats; how do these counts constrain the possible sequences? These and other related results will be published elsewhere.

Mathematical analyses of dance vocabulary and phrasing are usually more easily accomplished for classical dance forms such as ballet, in which movements are limited in number and performed in a strictly prescribed manner. The human body is capable of so many positions, gestures, and movements that more free-form movement styles make careful analysis difficult. Mathematical analyses can facilitate the discovery of overlooked sequences that fit desired constraints, allow deeper understanding of typical movement patterns, and help develop mathematical techniques for analyzing dance.

References

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