Fisheye View of Tessellations

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Abstract

In this paper we provide a description of several methods for creating tessellations of the hyperbolic plane, and patterns obtained from them by allowing gaps and overlapping. All designs are created using Wolfram Mathematica package *Tess*, visualized in the Poincare and Klein disk models, hence the name Fisheye view.

Introduction

What is the shape of the universe? This question, that scientists still do not know how to answer, is related to determining if the universe behaves like hyperbolic, Euclidean or spherical space, respectively of constant curvature minus one, zero or plus one. The differences between these spaces can be illustrated using tessellations in Fig. 1. While there are finitely many regular tessellations of the sphere and Euclidean plane, tilings of the hyperbolic plane are much richer and diverse [1], both in the choice of tessellation and the visualization model [2, 3]. We focus on the Poincare and Klein disk models that can be thought of as providing mathematicians and artists with ”fisheye views” of the hyperbolic plane. In this paper, we present

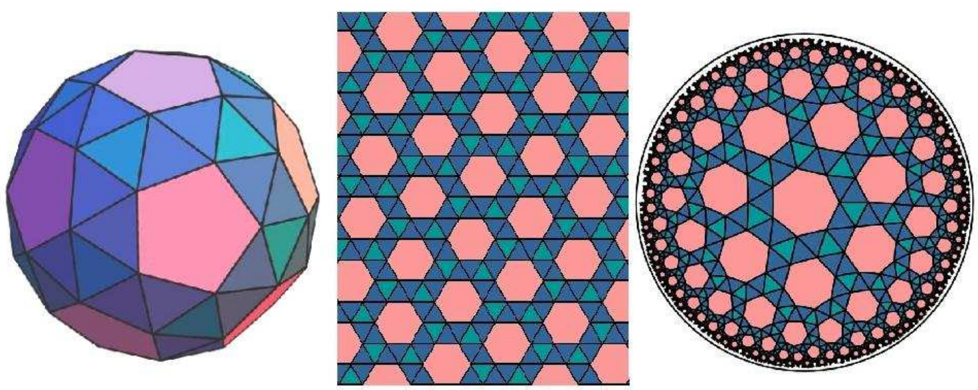

Figure 1 : *Spherical or elliptic (3,3,3,3,5), Euclidean (3,3,3,3,6) and hyperbolic (3,3,3,3,7) tessellations.*

and discuss tessellations created using Wolfram's Mathematica package *Tess* [4, 5]. *Tess* enables creating and drawing tessellations of Euclidean, elliptic and hyperbolic plane with regular polygons as tiles based on their Schläfli symbols[1]. Choice of Schläfli symbols can be arbitrary but each symbol may correspond to many, one or no tessellations at all. Basic tessellations can be enriched by introducing various patterns into the fundamental domain, and by relaxing the ”no overlap” and ”no gaps” conditions that normally go along with tessellations.

[1]Schläfli symbols determine a tessellation by specifying the polygons around each vertex of the tessellation. Notice that the correspondence between the symbol and the tessellations is one to many, i.e., the symbol does not necessarily define the tessellation uniquely.

Mathematics Within Art

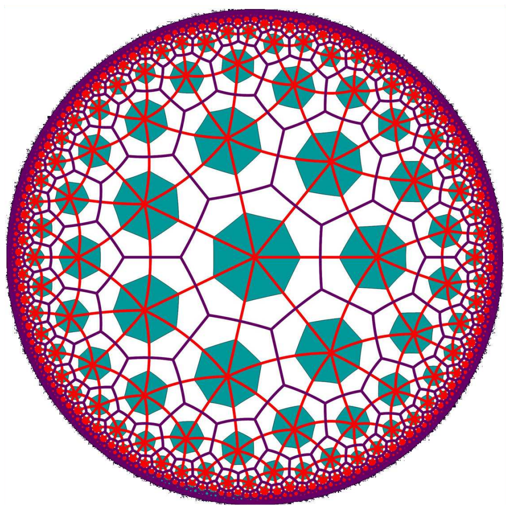

Figure 2 : Caught in a Dual Net *tessellation by R. Sazdanovic 2008.*

Mathematics, or more precisely geometry, provides the basis for the algorithm used to create tessellations, but also for ways of creating new from the existing ones and combining them. Dual tessellation is obtained by connecting the incenters of adjacent polygons in the original tessellation. *Caught in a dual net* [2], shown on Fig. 2, consists of three different tessellations: $(7, 7, 7)$ purple wire model, its dual tessellation determined by $(3, 3, 3, 3, 3, 3, 3)$ in red, and the hidden tessellation $(6, 6, 7)$ with turquoise heptagon and two hexagons bounded by wire models of previous two tessellations, their edges alternating between red and purple, visualized in the Poincare disk model. Colors reflect properties of the symmetry group of the tessellation: isometries map polygons of the same color to each other, but the color of each class can be chosen.

Art Within Mathematics

At the first sight, mathematics and art could not be more different: free-flowing creative expression versus rigid definition/proof analysis. Even if you gave this idea more thought it could be hard to imagine their coherent coexistence. Yet they are intertwined and examples are numerous: the Great Pyramid, the Parthenon, and the Colosseum, golden ratio, work of Dürer, Da Vinci, and famous 20th century artists M.C. Escher, R. Penrose, S. Dali.

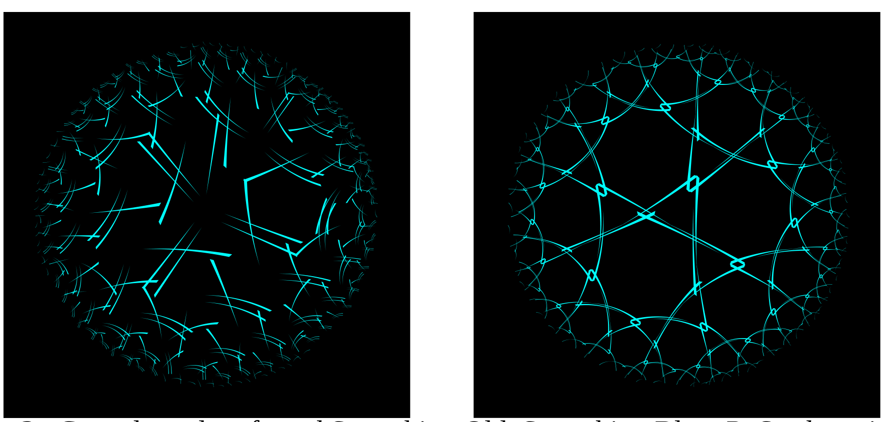

Figure 3 : Crystal coral reefs *and* Something Old, Something Blue, *R. Sazdanovic 2012*

In particular, the creative artistic component of drawing tessellations includes but it is not limited to the choice of color. Here is a step-by-step description of the whole process used for creating tessellations in this paper. First, all of them are based on simple tessellations determined by Schläfli symbols and constructed using computer program *Tess*. Second, we use additional code to determine the fundamental region of the tessellation. Once the shape and the size of the region is known, we can choose the basic pattern (motif) and hand it back to the computer to make copies of it according to the symmetries of the original tessellation in order to cover the whole plane.

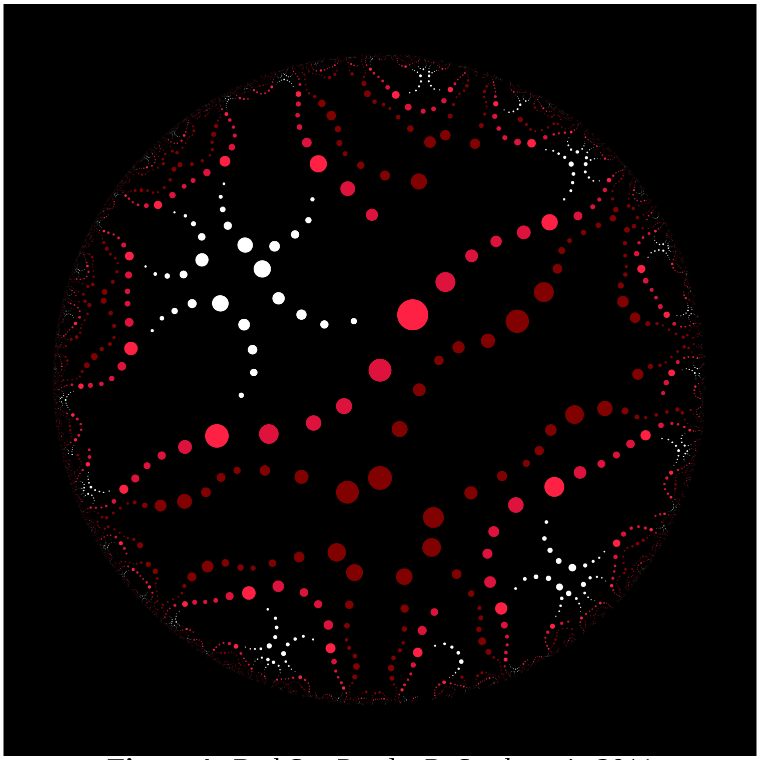

Figure 4 : Red Sea Pearls, *R. Sazdanovic 2011*

Digital print *Red Sea Pearls*, see Fig. 4, is based on the hyperbolic tessellation $(7, 7, 7, 7)$ and realized in the Poincare disk model. The core pattern consists of red and white circles of various sizes, and color intensities. It is extended to the whole hyperbolic plane under symmetries of the original tessellation, but the asymmetry of the pattern has the overall effect of breaking the original symmetry of the tessellation. Note that there are infinitely many tessellations of the hyperbolic plane: all of them can be used for creating aesthetically pleasing graphics. What is very interesting is that choosing a beautiful pattern does not guarantee that the final result will be pretty. It is often very hard or even impossible to predict the final tessellation based on the pattern and the symmetries of the original tessellation.

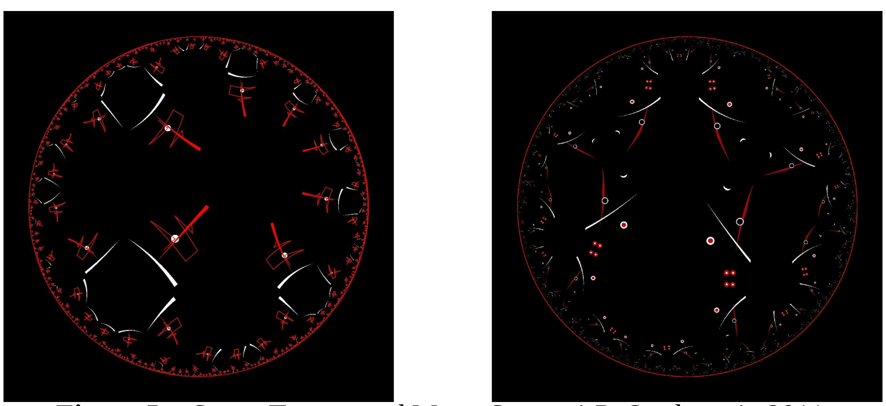

Figure 5 : Seven Towers *and* Moon Samurai *R. Sazdanovic 2011*

Tessellations *Seven Towers* and *Moon Samurai*, Fig. 5, are inspired by Japanese culture. Basic patterns

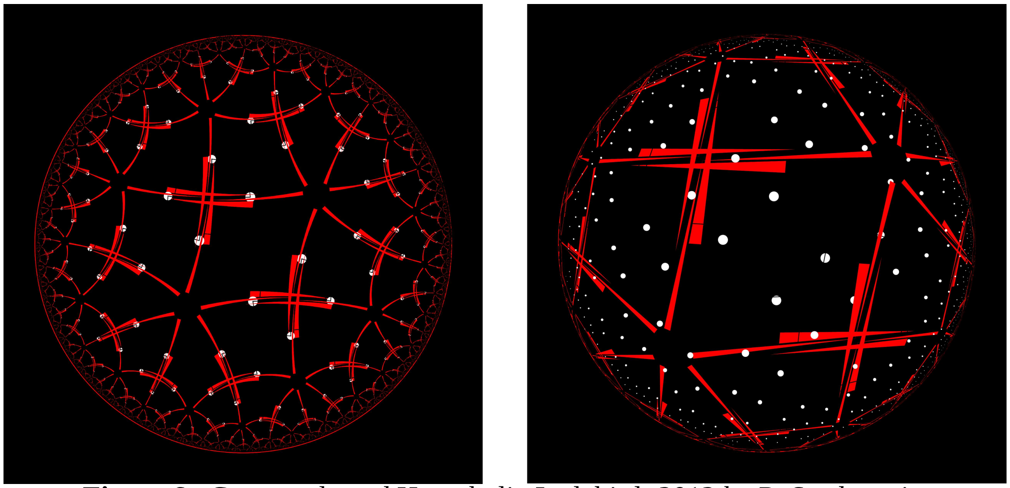

Figure 6 : Crossroads *and* Hyperbolic Ladybird*, 2012 by R. Sazdanovic*

are chosen after a series of not–so–successful tries, in such a way that the final designs contain structures that resemble pagodas for the one on the left, and Japanese samurai on the right. Both tessellations are realized in classical black, red and white color scheme on the black background, emphasizing local seven-fold and six-fold symmetry. Using the same color scheme, we have constructed *Crossroads* and *Hyperbolic Ladybird*, shown on Fig. 6, based on the Schläfli symbol $(4, 4, 4, 6)$ with almost identical patterns. The difference between them comes from the realization model: one was visualized in Poincare and the other in the Klein disk model.

All of the tessellations described above are evidence of the intricate two-way relations between mathematics and art. Geometry and combinatorics behind the tessellations provide the framework and limitations for the creative, artistic expression. The other way around, choice of the basic motif can alter and hide the mathematical structure, depending on its internal symmetries and the way it is positioned inside the fundamental region. The effects of weaving the basic pattern into the original tessellation can be subtle or dramatic, with the additional richness coming from the choice of the visualization model: Poincare or Klein disk. In conclusion, the idea that visual artwork can be used to convey and alter the geometric elegance of mathematical structures associated with a Schläfli symbols that are merely a sequence of numbers is truly fascinating.

References

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