Mohr or Mascheroni?

Elaine Krajenke Ellison
Retired from West Lafayette High School, Purdue University
5739 Wilena Place
Sarasota, Florida 34238
E-mail: eellisonelaine@yahoo.com
www.mathematicalquilts.com

Abstract

The cardioid has interested mathematicians for years. Investigating the cardioid along with other Mascheroni constructions became a fascinating adventure for me and for my classroom students! The following paper shows eleven constructions that can be used in the high school or college classroom to study mathematics along with inspiring fascinating designs. The unusual constructions will add interest, variety, and challenge to students.

Introduction

Unappreciated Danish mathematician Georg Mohr (1640-1697) published an unusual book in 1672 entitled Euclides Danicus. In this book, Mohr showed that any pointwise construction that can be performed with compass and straightedge (i.e. any "plane" problem) can be done with a moveable compass alone. As a compass can't draw a straight line, we need to know that two distinct points in Euclidean geometry do determine one unique line. This makes the use of a straightedge superfluous. (For 125 years this interesting discovery was overlooked until Lorenzo Mascheroni (1750-1800) may have discovered Mohr's work.)

Over one hundred years later, in his 1799 tractate The Geometry of the Compasses, Mascheroni showed that every ruler-and-compass construction could be accomplished by a compass alone. Interesting enough, this book was dedicated to Napoleon Bonaparte.

Mascheroni's parents were Maria Ciribelli and Paolo Mascheroni del Il'Olmo. Paolo was a wealthy landowner. Lorenzo was educated with the idea of becoming a priest. Ordained at 17, he taught rhetoric and further taught physics and mathematics at the seminary in Bergamo, Lombardy. In 1786, Mascheroni became professor of algebra and geometry at the University of Pavia. From 1789 to 1791 he was head of the Academia degli Affidati.

For his excellent contributions, Mascheroni received a number of honors such as election to the Academy of Padua, the Royal Academy of Mantua and the Societa Italiana delle Scienze. He was appointed as a deputy in the governing legislative assembly in Milan in 1797. The French had been working on the introduction of the metric system of weights and measures and on April 7, 1795 the National Convention had passed a law introducing the metric system, putting mathematician Legendre in charge of the transition to the new system. Mascheroni was sent to Paris to study the new system and to report to the governing body in Milan. He published his report in 1798, but the War of the Second Coalition began in 1799 while Napoleon Bonaparte was in Egypt and the French government was in crisis. Austria took the opportunity to secure Russia as an ally and then attack the French on several fronts including the north of Italy. Mascheroni was unable to return to Milan due to the war and the Austrian occupation of the city in

1799. He remained in Paris, where he died in the following year after a brief illness resulting from complications after catching a cold.

From the time of Mascheroni's work "compass only" constructions were known as Mascheroni constructions. In 1928, a copy of Mohr's book was found in a Copenhagen bookstore by Hjelmslev, a mathematician. Had Mascheroni seen Mohr's work, or were his constructions original? No one seems to know. Perhaps "Mohr constructions" would be a more accurate name for the compass-only constructions. Or, giving both men their due, "Mohr-Mascheroni constructions" would be a fitting name.

Assumptions

We need some starting points for Mohr and Mascheroni's constructions. Included in the list of requirements are:

1. Two distinct points determine a line
2. One must be able to draw a circle with a given center and radius.
3. One must be able to find the point or point of intersection of two circles.
4. One must be able to find the point(s) of intersection of a straight line and a circle.
5. Finding the point of intersection of two given straight lines is essential.

Construction #1

Goal: Find a point E on $\overrightarrow{AB}$ so that AE = 2(AB).

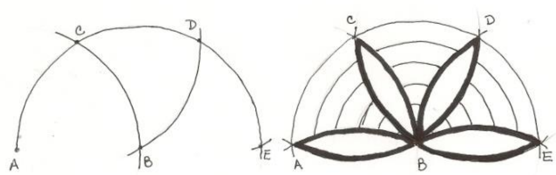

Figure 1: *Construction #1 and design*

Analysis: Using $\overline{AB}$ as a unit length, construct arc (B, AB). Draw (A, AB) with the intersection at C. From C, arc (C, AB) labeling the intersection at D. Then at D, arc (D, AB) intersecting (B, AB) at point E. The segments $\overline{AB}$, $\overline{AC}$, $\overline{BC}$, $\overline{CD}$, $\overline{BD}$, $\overline{DE}$, and $\overline{BE}$ are all congruent. These segments form equilateral ΔABC, ΔCBD, and ΔBDE. Each of the angles <ABC, <CBD, <DBE is 60°. This makes $\overleftrightarrow{ABE}$ a straight line. To prove that E lies on $\overleftrightarrow{AB}$, a proof by contradiction can be used.

Construction #2

Goal: Construct a line segment whose measure is n times the measure of a line segment, where n = 1, 2, 3,…..n

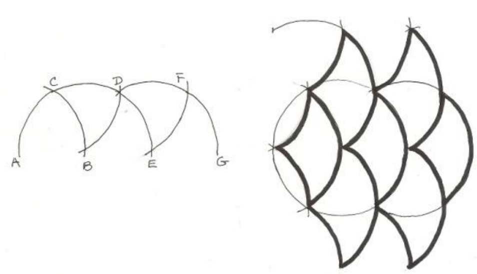

Figure 2: *Construction #2 and design*

Analysis: Using the strategy used in construction #1, we will extend the same procedure for AG = 3(AB). From there we will generalize the procedure for any n such that n(AB) where $\overline{AB}$ is a unit segment. From E, swing arc (E, AB) such that F is located. At F, swing a full arc AB such that this arc crosses at point G. Similar reasoning can be applied to prove that A, B, E, G, ….are all collinear.

Construction #3

Goal: Construct a line segment whose measure is $\frac{1}{n}$ th the measure of a given line segment, n≥2.

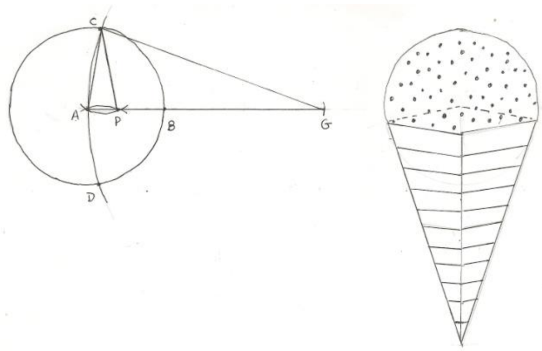

Figure 3: *Construction #3 and design*

Analysis: We will illustrate constructing the segment $\frac{1}{n}$th the length using n=3. This strategy can be applied to n≥2, so n can be any whole number greater than or equal to 2. As in construction #2, we construct AG = 3AB. Draw circle (A, AB). Draw arc (G, GA) intersecting circle (A, AB) at points C and D respectively . Arc (C, CA) and (D, DA) until they cross at A and P. P is the trisection point of AB or AP = 1/3 AB.

Analysis: Point A, P, G lie on the perpendicular bisector of $\overline{CD}$. $\overline{CD} \cong \overline{GD}$, $\overline{CP} \cong \overline{DP}$, and $\overline{DA} \cong \overline{CA}$. A, P, G are collinear. Then ΔCGA is similar to Δ PCA by AA. AP/AC = AC/AG. Because $\overline{AC} \cong \overline{AB}$, AP/AB = AB/AG. AG = 3(AB) or AB/AG = 1/3. By substitution AP/AB = 1/3, or AP = 1/3(AB).

Construction #4

Goal: Construct the midpoint of a line segment specified by two distinct points A and B.

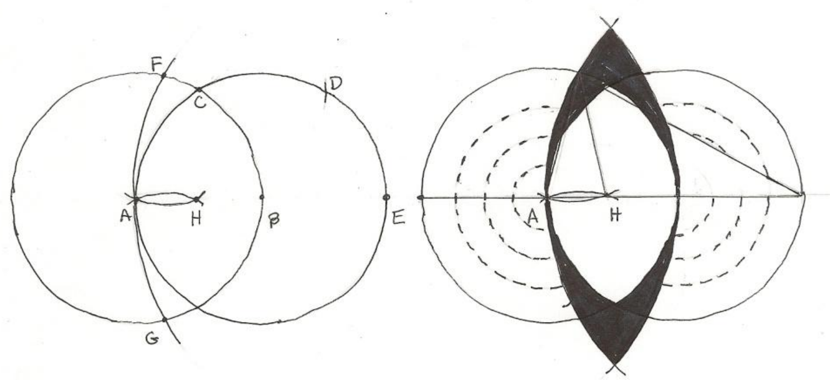

Figure 4: *Construction #4 and design*

Construct two circles of radius $\overline{AB}$ from A and point B on circle A. Label the intersection of the two circles C and I respectively. Using the unit length of AB, arc (C, AB) using circle B. Label point D where the arc crosses circle B. Using AB again, arc (D, AB) and label the intersection point E. From construction #1, $\overleftrightarrow{ABE}$ are collinear and AE = 2AB. Now reset the compass to the length of $\overline{EA}$. Swing arc (E, EA) and label the crossing with circle A as points F and G. From F and G, arc (F, FA) and (G, FA). The arcs cross at Point A and point H. H is the midpoint of AB.

Analysis: Show that ΔAFH is similar to ΔAFE by AA. Then AF=½ AE and AH = ½ AB

Construction #5

Goal: Find the center of a circle with a compass and no straightedge. Mascheroni's method of finding the center of a circle with compass alone is beyond the scope of this paper. However, another construction of unknown origin is shown here.

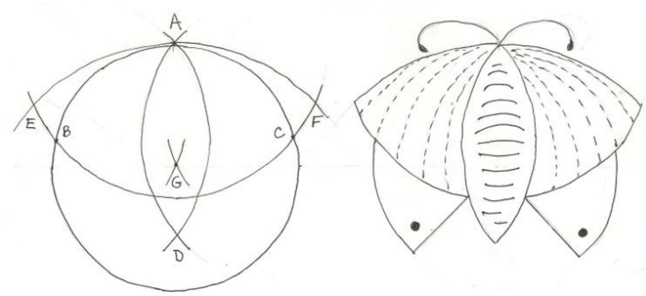

Figure 5: *Construction #5 and design*

The circle is given. From any point A on the circle, swing an arc crossing the circle at points B and C. With r = AB, using B and C as centers, draw arcs that intersect at point D. D can be inside or below the circle, depending on the length of the first arc. Using $\overline{AD}$ and D, draw arcs to intersect at points E and F. Using $\overline{AE}$ as a unit length and points E and F, draw arcs intersecting at G. G is the center of the circle.

Analysis: $\overline{AB} \cong \overline{AC}$. ΔABD ≅ ΔACD. Now ΔAED ≅ ΔAFD and ΔAEG ~ΔADE. Points A, G, and D are collinear. ΔBAG≅ ΔCAG and $\overline{BG} \cong \overline{GC}$. Notice also that the points A, B, and C all lie on the same circle, so segment $\overline{BG} \cong \overline{AG} \cong \overline{GC}$. For the proof by inversion geometry, see L. A. Grahams *The Surprise in Mathematical Problems* (Dover, 1968) problem #34.

Construction #6

Goal: Construct the circumcircle of ΔABC with compass alone. This construction is left unjustified. See [8].Given the points A, B, and C, construct the circumcircle of ΔABC. Begin by using the vertex B and the length of $\overline{BA}$ to draw a circle. Then using vertex C, draw a circle with the radius of the length of $\overline{CB}$. Label the intersection of the circles point D on the left, point E on the right. Using vertex A and the length of $\overline{AB}$, swing a circle. Using D as the center of the circle, swing a circle using the length of $\overline{DB}$. Using E as a center of a circle, swing the circle of length $\overline{EB}$. Label the intersection of the circle centered at D and the circle centered at E with point F. Using point F and a radius the length of $\overline{FB}$. Label the intersection of the circle centered at A and the circle centered at F point G. With point G and the radius of length of $\overline{GB}$, swing a circle. Label the intersection of the circle centered at point B with point H. Label the intersection of the circles with center G and center B as point K. Using the center H and a radius of the length of $\overline{HB}$, swing a circle. Further, using center K and a length of $\overline{KB}$, swing a circle.

Where the circles of centers H and K intersect, label point L. Now, using $\overline{LA}$, or $\overline{LB}$, or $\overline{LC}$, swing the circumcircle of ΔABC.

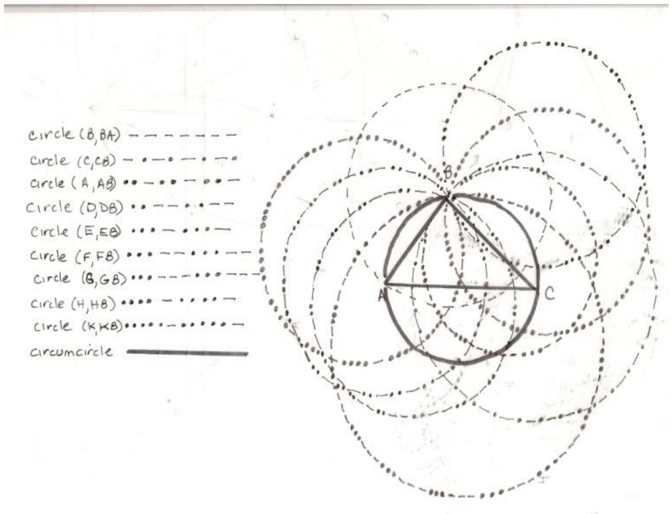

Figure 6: *Construction #7 as a design*

Construction #7

Goal: Construct a line parallel to a given line $\overleftrightarrow{AB}$ and through any point P outside the line.

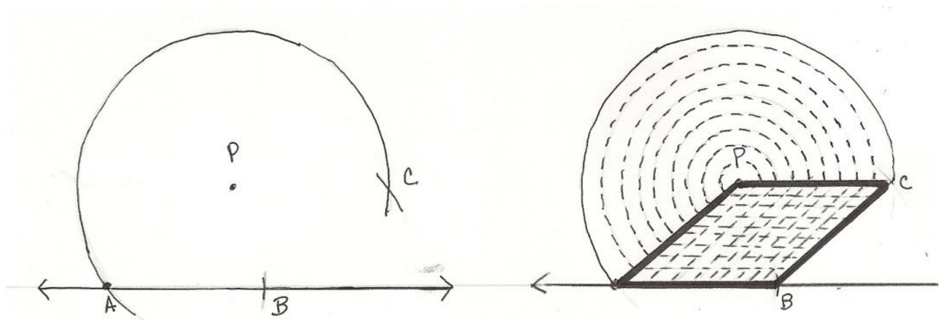

Figure 7: *Construction #7 and design*

Essentially the corners of a rhombus are constructed in order to find the missing parallel line. Given two points A and P plus a line through point A. Using point P and the length of $\overline{PA}$, swing a very full arc. From point A, locate a point B on the given "line" so that $\overline{PA} \cong \overline{AB}$. From point B, arc $\overline{PA}$ until it crosses the original large arc from point P. Label point C. ABCP forms a rhombus.

Analysis: Because the rhombus has opposite sides parallel, we now have the required parallel.

Construction # 8

Goal: Through a point P (not on $\overleftrightarrow{AB}$) construct a perpendicular to $\overleftrightarrow{AB}$.

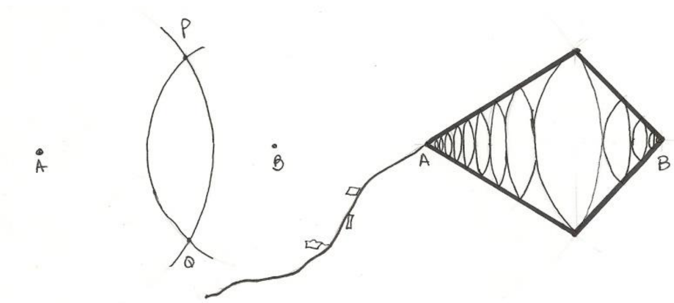

Figure 8: *Construction #8 and design*

Draw a full arc (A,AP) and (B,BP). Label the lower point of intersection of the two arcs point Q. Then $\overleftrightarrow{PQ} \perp \overleftrightarrow{AB}$.

Analysis: $\overleftrightarrow{AB}$ and $\overleftrightarrow{PQ}$ intersect at C. ΔAPB and ΔAQB are congruent by SSS. Using the corresponding parts of the congruent triangles <PAB and <QAB are congruent. $\overline{AC} \cong \overline{AC}$. Then ΔACP and ΔACQ are congruent by SAS. Further <PCA ≅ <QCA and on a straight line making the angles at this point right angles and then the lines are perpendicular to each other.

Construction #9

Goal: Locate the points of intersection of a straight line given by two points A and B, and a given circle, (O, r). There are two cases for us to consider here. See [1] page 80.

Case 1: Where the center of the circle does not lie on the given line. Using the figure below with (O, r) and the straight line $\overleftrightarrow{AB}$. Find Q, the point of intersection of arcs (B, BO) and (A, AO). Draw the circle (Q, r). The points of intersection of circle (O, r) and (Q, r) are P and R, P being on the left-sided intersection.

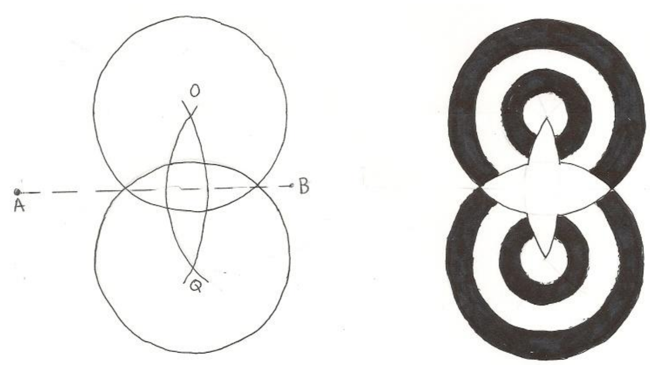

Figure 9: *Construction #9a and design*

Analysis: Point Q was chosen so as to make $\overleftrightarrow{AB}$ the perpendicular bisector of $\overline{OQ}$. O and Q are equidistant from P and R. By drawing a circle (Q, r) congruent to and intersecting circle (O, r), the common chord $\overline{PR}$ is also the perpendicular bisector of $\overline{OQ}$.

Case 2: Where the center of the circle lies on the given line. Using circle (O, r) and the straight line $\overline{AB}$

as shown here. Construct a circle (A, x) so that x is large enough to intersect (O, r) in two points S and T. Using congruent triangles SRO and TRO, and ΔSOP and ΔTOP, we can show that P and R are the midpoints of both major and minor arc ST. Points P and R are the required points of intersection of circle (O, r) and $\overleftrightarrow{AB}$.

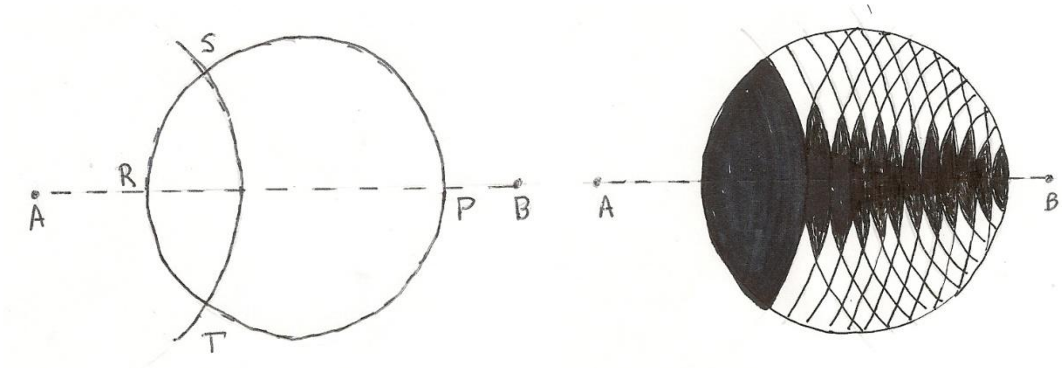

Figure 10: *Construction #9b and design*

Construction #10

Goal: Given two adjacent corner points of a square, find the other two points.

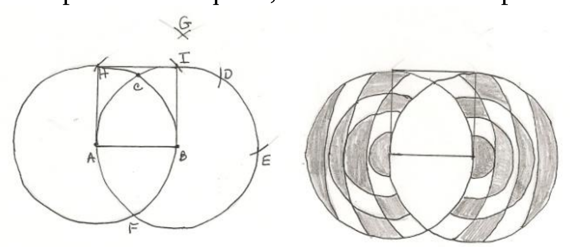

Figure 11: *Construction #10 and design*

Given two points A and B of a square. Draw circles (A, AB) and (B, AB). Label intersections C above intersection and point F the lower intersection. Using $\overline{AB}$, arc (C,AB). Label the intersection with the circle B as point D. From point D, arc (D, AB) and label point E. Open compass to the length of $\overline{CF}$. Using A and E as centers, draw the two arcs that intersect at G, using CF as a radius. With radius $\overline{GB}$ and centers A and B, draw the arcs that cut the circles at H and I. Points H and I are the other two corners of the square.

Analysis: Let AB = 2x. Then the length of CF= 2x√3. Using Pythagorean Theorem, BG=2x√2, As BG is swung from point A and B, crossing the respective circles at points H and I, 2x√2 would be the correct length of the diagonals of the required square.

Construction #11

Goal: Given a circle and center, find the four corners of an inscribed square using compass only. This problem has a fascinating history as it relates to Napoleon. Napoleon had Mascheroni working with his generals and warfare usage of compass construction. So the term "Napoleon's problem" makes sense.
Another way of stating the problem is to divide a circle into 4 equal arcs.

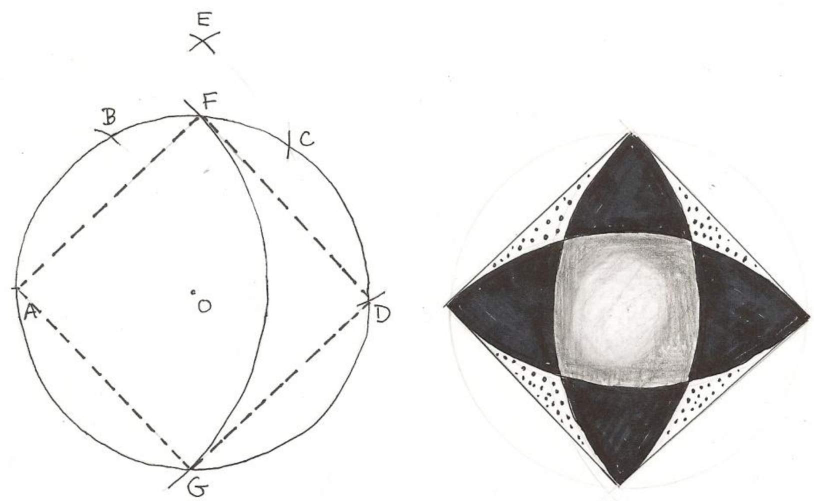

Figure 12: *Construction #11 and design*

Using circle O, label point A at any point on the circle. Using the circle's radius of $\overline{OA}$, arc (A,OA), (B,OA) , and (C,OA). Adjust the compass to the length of $\overline{AC}$. With point A arc (A,AC) and (D,AC). Label the intersection point with E. Using radius $\overline{OE}$ and point A, draw a large arc that cuts the circle at points F and G. Now A, F, D, and G are the corners of an inscribed square.

Analysis: This construction is easily justified using simple algebra. Let AO = 2x, then the length of AC = 2x√3. Then OE= 2x√2. Thus AF = 2x√2 and the side of the square

Conclusion

The Mohr-Mascheroni constructions and designs give the reader a fascinating start to developing fine art blending mathematics, geometry, and history together in a pleasing manner!

References

[1] Alfred S. Posamentier and William Wernick, *Advanced Geometric Constructions*, Dale Seymour Publications, Palo Alto, California, p. 75-84. 1973.

[2] Carl B. Boyer, *A History of Mathematics*, John Wiley and Sons, New York, p.406, 575. 1968.

[3] Martin Gardner, *Mathematical Circus*, Knopf, New York, p. 216-231. 1979.

[4] Harold R. Jacobs, *Mathematics—A Human Endeavor*, W. H. Freeman and Co., San Francisco, p. 195-200. 1970.

[5] Jurgensen, Brown, Jurgensen, *Geometry*, Houghton Mifflin, Boston, p. 410. 1992.

[6] Dan Pedoe, *Geometry and the Visual Arts*, Dover Publications, Inc., New York, p. 22-25. 1976.

[7] J. J. O'Connor & E. F. Robertson, http://www.gap-system.org/~history/biographies/Mascheroni.html.

[8] http://gogeometry.com/geometry/mascheroni_compass_construct_circumcircle.htm.