Exploring the Projective Plane via Variations on the Faceted Octahedron

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Abstract

We explore the topology of the projective plane and of its immersions in 3-space via two models based on the faceted octahedron. This is a symmetrical immersion closely connected to a smooth immersion discovered by Werner Boy.

Introduction

Unlike the Klein bottle, the projective plane is difficult to model so that the topology is easy to visualize. A smooth immersion in 3-space must contain at least one triple point where three different parts of the surface all intersect [4]. A satisfying and symmetrical *topological* immersion is the faceted octahedron or Rheinhardt heptahedron, a seven-faced uniform polyhedron in the shape of an octahedron with four triangles missing and replaced by three squares that all intersect at a triple point exactly at the center (Figure 1, right). The one flaw in this model is that the "figure 8" arrangement of the four faces that meet at each vertex is an obstruction to deforming the model into a *smooth* immersion. In "What is ... Boy's surface" (2007) [4], Rob Kirby observes that these singular points can be removed by introducing a loop in the surface along three of the edges connecting the six vertices in pairs. We discovered this solution, ourselves, while examining the faceted octahedron and other non-convex uniform polyhedra. This is in contrast to the popular cross-cap, which has a simpler construction, but has the disadvantage that the singular points are not so easily untangled. For models of the cross-cap, see [3], [5] and even [2] in its implementation. As we note below, the process of forming a faceted octahedron from a half cubotahedron can also lead directly to this smooth immersion of the projective plane in 3-space equivalent to the one presented by Werner Boy, himself, in his dissertation [1].

The Models

A Fabric Model. The cuboctahedron shown in Figure 1 was constructed from stiff, non-elastic fabric materials and contains zippers that separate it into two congruent halves. One of the halves contains some extra cuts that allow it to be reformed into the heptahedron and re-zipped to itself (for the most part) in the required way. One of the theee zippers is, for practical reasons, along the diagonal of one of the squares, so the re-zipping is obstructed by the fact that another square must pass through the same line in 3-space.

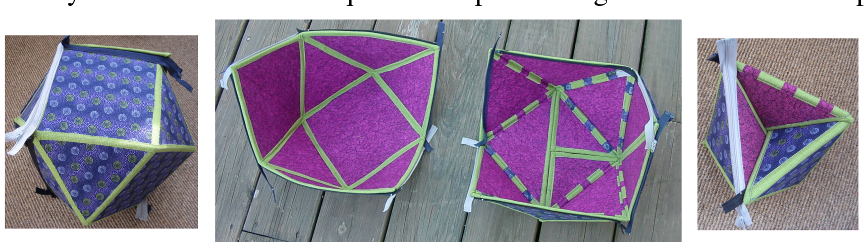

Figure 1 : *The fabric model: zipped to its other half, then separated, re-configured and zipped to itself.*

Figure 2 shows the re-configurable half of our fabric construction laid out in the form of a net along with a diagrammatic version. The second diagram is an alternative net where the cuboctahedron has been divided along polygonal edges forming a planar hexagon. The second version would be more satisfying from a mathematical point of view.

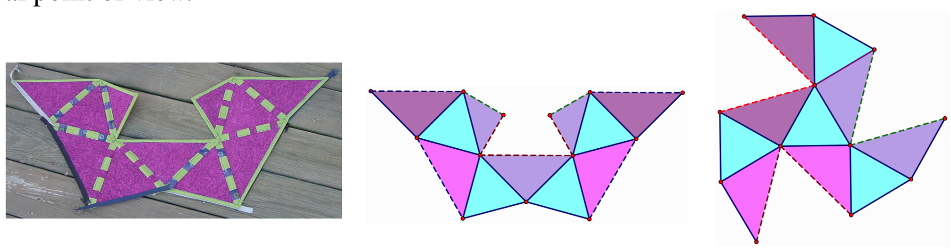

Figure 2 : *Fabric model laid out, the net, and an alternative net.*

A Virtual Model. By following the transition from the half-cuboctahedron to the faceted octahedron in this virtual model, we see how the loops in the surface arise. These loops along a line of self-intersection lead to a smooth immersion. Without the extra cuts, the path of deformation is forced in this direction. Figure 3 shows the initial, intermediate and final stages of the deformation. The three triangles along the boundary are twisted 180° to the right and the adjoining squares are stretched, with narrow triangles and rectangles attached to their boundaries to avoid the creases and singular points. The two highlighted edges in the upside-down view are to be joined in the final stage. The triangle adjacent to the longer highlighted edge has been removed for better viewing, as has one narrow rectangle in the final stage. The final configuration reveals the continuous line of self-intersection looping through the triple point three times, as required by a smooth immersion. The authors can provide coordinates and/or Mathematica files for anyone wishing to explore this model further. One interesting direction for study might be the problem of turning a sphere inside-out without leaving a crease. Kirby's article [4] points out that any smooth immersion of the projective plane holds the key to such an eversion of the sphere.

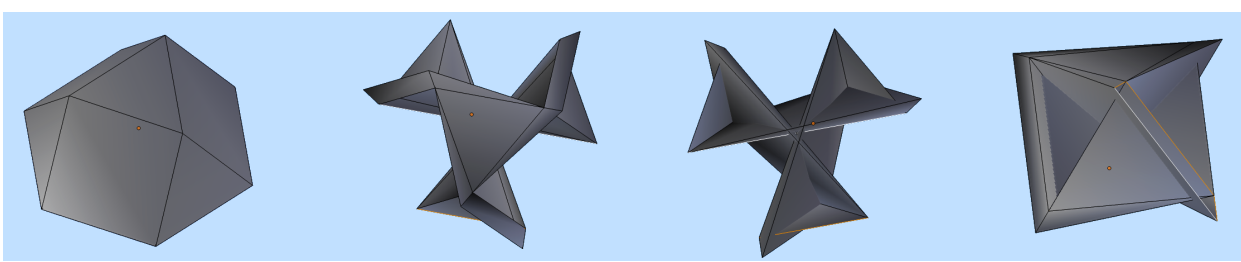

Figure 3 : *Virtual model in its initial, intermediate and final stages.*

References

[1] W. Boy, "Über die Curvatura integra und die Topologie geschlossener Flächen", *Math. Ann. 57* (1903), 151–184, `http//:surfaces.gotfork.net/boy.pdf`.

[2] S. Goldstine, "Fortunata's Purse", In S. Belcastro & C. Yackel (Eds.), *Making Mathematics with Needlework*, Wellesley, Massachusetts: A K Peters, Ltd (2008).

[3] C. Irving, "Making the real projective plane", *The Mathematical Gazette 89 no. 516* (2005), 417– 423.

[4] R. Kirby, "What is . . . Boy's Surface", *Notices of the AMS 54* (2007, November), 1306–1307.

[5] E. Peters, "A Knitted Cross-cap", In S. Belcastro & C. Yackel (Eds.), *Crafting by Concepts*, Natick, Massachusetts: A K Peters, Ltd. (2011).