Extension of Neo-Riemannian *PLR-group* to Seventh Chords

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Abstract

This paper extends the concept of the *PLR-group* from the neo-Riemannian theory, which acts on the set of major and minor triads, to a *PS-group*, which acts on the set of major and minor seventh chords. Like the PLR-group, the PS-group is isomorphic to D_{12}, the dihedral group of order 24, and only compositions of two operations are needed to generate all group elements. Unlike the actions from PLR-group, there are only two operations that both preserve the closure of the set of major and minor seventh chords, as well as map three out of four notes from a chord to a resulting image.

Introduction

Mathematical concepts and structures can be found in many places within music. For example, any two pitches that differ from one another by an integral number of octaves sound alike, and as such form an equivalence class of pitches. In Western music, we can use the equal-tempered tuning to divide an octave into twelve pitch classes, with the difference of a half-step or a semitone between two consecutive pitch classes. Furthermore, we can map these twelve pitch classes to integers modulo 12, commonly referred to as Z_{12}, and start by mapping a pitch class C to number 0. Figure 1 shows the twelve pitch classes commonly used in Western music, as well as the bijective map of the pitch classes into Z_{12}, which enables us to use algebraic concepts to model common musical events [1-3].

A	A# (B♭)	B	C	C# (D♭)	D	D# (E♭)	E	F	F# (G♭)	G	G# (A♭)	A

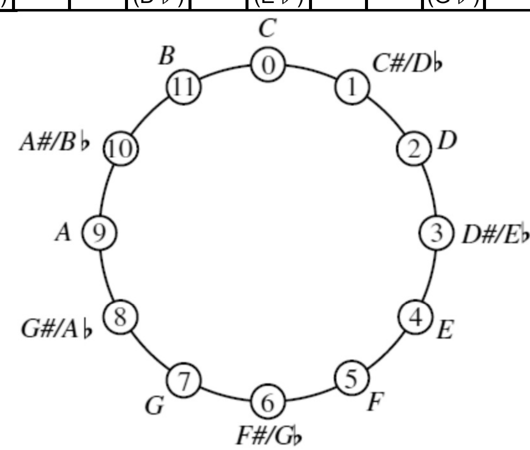

Figure 1: *Top: Octave division into 12 pitch classes under equal tempered tuning. Bottom: Mapping pitch classes to integers modulo 12, as described in [2] (taking 0 to be C).*

A triad is a set of three distinct pitch classes, and will be denoted as a triple <a, b, c>. That is, a triad is a chord consisting of three notes all sounding at the same time. Figure 2 shows consonant triads, which have characteristic lengths of intervals between pitch classes a, b, and c, and thus are perceived pleasantly when played together. For instance, a minor triad consists of its root note, played along with notes that are three and seven semitones above the root note. From an algebraic standpoint, we can now apply transformations $T_n, I_n: Z_{12} \rightarrow Z_{12}$ to the set of consonant triads, where $T_n(x) = (x+n) \, mod \, 12$, and $I_n(x) = (-x+n) \, mod \, 12$. Essentially, the first map transposes each pitch class by n semitones, while the second map can be thought of as a reflection of a pitch class about the axis represented by a line connecting nodes 0 and 6 in the map of Figure 1. As a demonstration of these concepts, consider the C-major triad $\langle 0,4,7 \rangle$. Then $T_2\langle 0,4,7 \rangle = \langle 3,7,10 \rangle$, and $I_2\langle 0,4,7 \rangle = \langle 2,10,7 \rangle$. The remaining possibilities are depicted in Figure 2, which was adopted from [2] and which shows the sets of both major and minor triads. If the first entry is counted as a zero, then the nth entry in the left column is

$$T_n\langle 0,4,7 \rangle = \langle T_n(0), T_n(4), T_n(7) \rangle$$

and the nth entry in the right column is

$$I_n\langle 0,4,7 \rangle = \langle I_n(0), I_n(4), I_n(7) \rangle.$$

Major Triads	Minor Triads
C = <0, 4, 7>	<0, 8, 5> = f
C♯ = D♭ = <1, 5, 8>	<1, 9, 6> = f♯ = g♭
D = <2, 6, 9>	<2, 10, 7> = g
D♯ = E♭ = <3, 7, 10>	<3, 11, 8> = g♯ = a♭
E = <4, 8, 11>	<4, 0, 9> = a
F = <5, 9, 0>	<5, 1, 10> = a♯ = b♭
F♯ = G♭ = <6, 10, 1>	<6, 2, 11> = b
G = <7, 11, 2>	<7, 3, 0> = c
G♯ = A♭ = <8, 0, 3>	<8, 4, 1> = c♯ = d♭
A = <9, 1, 4>	<9, 5, 2> = d
A♯ = B♭ = <10, 2, 5>	<10, 6, 3> = d♯ = e♭
B = <11, 3, 6>	<11, 7, 4> = e

Figure 2: *The set S of consonant triads.*

It is interesting to note that these two maps T and I, when applied to the pitch classes of consonant triads, form an algebraic group which is isomorphic to the dihedral group D_{12} of order 24, the symmetry group of a 12-sided regular polygon. The group formed by maps T and I is also known in literature as the *T/I-group* and its properties have been thoroughly investigated [2, 3].

Recent work from neo-Riemannian theory has focused on so-called *PLR-group*, and like the T/I-group, the PLR-group acts on the set of consonant triads. Neo-Riemannian theory was originally established by Hugo Riemann [5], a music theorist of the late 19th century, who developed a framework that related specific triads to one another. The work of Lewin [4] continued to explore this framework and developed an alternate mechanism to describe the relationship among triads via the PLR-group, while the work in [3] generalized pitch class mappings as *uniform triadic transformations*. The *PLR-group* consists of three maps P, L, and R, which can be thought of as *parallel*, *leading tone exchange*, and *relative* inversions respectively. With respect to the consonant triads depicted in Figure 2, the P map switches the first and the third note of a triad, the L map switches the last two notes, while the R map switches the first two notes of a triad. For example, $P\langle 0,4,7 \rangle = \langle 7,3,0 \rangle$, $L\langle 0,4,7 \rangle = \langle 11,7,4 \rangle$, and $R\langle 0,4,7 \rangle = \langle 4,0,9 \rangle$. All three maps change the type of a triad, so that a major triad maps to a minor one, and *vice versa*. Work in [2] illustrated that the PLR-group is isomorphic to the dihedral group D_{12}, and that groups *T/I* and *PLR* are dual. The next section extends the concept of such mapping to the set of major and minor seventh chords.

The *PS-group*

We now consider a set *H* consisting of all major and minor seventh chords, illustrated in Figure 3. Seventh chords are extensions of triads introduced in the previous section and are formed by adding a fourth note to a triad, at the interval of a third above the fifth of the chord. Although the set of major and minor seventh chords is an extension of the set of consonant triads depicted in Figure 2, there are some significant differences among the two sets, particularly under actions of the T/I-group on the set *H*. For instance, the *I* operation maps triads to triads of the opposite type (e.g., a major to a minor triad), while the same operation maps seventh chords to chords of the same type. In addition, it appears that at least two generator seventh chords are necessary to generate the set *H* of major and minor seventh chords, unlike a single generator in the case of the consonant triads.

Major Seventh	Minor Seventh
C = <0, 4, 7, 11>	<0, 9, 5, 2> = d
C♯ = D♭ = <1, 5, 8, 0>	<1, 10, 6, 3> = d♯ = e♭
D = <2, 6, 9, 1>	<2, 11, 7, 4> = e
D♯ = E♭ = <3, 7, 10, 2>	<3, 0, 8, 5> = f
E = <4, 8, 11, 3>	<4, 1, 9, 6> = f♯ = g♭
F = <5, 9, 0, 4>	<5, 2, 10, 7> = g
F♯ = G♭ = <6, 10, 1, 5>	<6, 3, 11, 8> = g♯ = a♭
G = <7, 11, 2, 6>	<7, 4, 0, 9> = a
G♯ = A♭ = <8, 0, 3, 7>	<8, 5, 1, 10> = a♯ = b♭
A = <9, 1, 4, 8>	<9, 6, 2, 11> = b
A♯ = B♭ = <10, 2, 5, 9>	<10, 7, 3, 0> = c
B = <11, 3, 6, 10>	<11, 8, 4, 1> = c♯ = d♭

Figure 3: *The set H of major and minor seventh cords.*

Consider two mappings $P, S : H \to H$ so that

$$P\langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \rangle \ = < \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{a}, [(type\langle a, b, c, d \rangle) * 2 + \boldsymbol{d}] \ mod \ 12 >,$$
$$S\langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \rangle \ = < [(-1) * (type < a, b, c, d >) * 2 + \boldsymbol{a}] \ mod \ 12, \boldsymbol{d}, \boldsymbol{c}, \boldsymbol{b} >,$$

where the function *type(t)* returns either 1, when its argument *t* is a minor seventh chord, or a -1, when its argument *t* is a major seventh chord. In other words, the map *P* is a *prefix reversal* that essentially reverses the first three pitch classes of a seventh chord in Figure 3, and switches the last pitch class so that the resulting seventh chord is in set *H*, but it has the opposite type. For example, $P(G) = P\langle 7{,}11{,}2{,}6 \rangle = \langle 2{,}11{,}7{,}4 \rangle = e$. Similarly, the map *S* is a *suffix reversal* that flips the last three pitch classes of a seventh chord and switches the first pitch class so that the resulting chord is of opposite type. For example, $S(G) = S\langle 7{,}11{,}2{,}6 \rangle = \langle 9{,}6{,}2{,}11 \rangle = b$. Note that one can think of the *P* operation as switching the first and the third pitch class in a seventh chord instead of a prefix reversal of the first three pitch classes, since the second pitch class remains fixed by the map. Similarly, the *S* map can be thought of as exchanging the second and the fourth pitch classes in place of a suffix reversal, as the third pitch class remains fixed by the *S* map.

The *P* and *S* maps are similar to the *L* and *R* functions from neo-Riemannian theory, in a sense that starting with the C-major seventh chord and applying first *P* and then *S* maps, we obtain the same chord progression of different 24 major and minor seventh chords as we would when applying first *L* and then *R* functions on consonant triads. These two maps form the *PS-group*, which is obviously closed under composition operation. Each map is its own inverse, and the two maps are associative implicitly. We utilize a similar approach as in [2] to show that the maps P and S form a group that is isomorphic to the dihedral group D_{12} of order 24.

Theorem: *PS-group is isomorphic to the dihedral group D_{12} of order 24.*

Proof: The dihedral group D_{12} of order 24 is specified by the following relation:

$$D_{12} = \{x, y \mid x^2 = 1, y^{12} = 1, (x\,y)^2 = 1\}.$$

Let $x=P$ and $y=PS$. Since, as stated above, the orbit of C-major seventh chord under PS composition has 24 elements, $(PS)^{12}=1$. Furthermore, since each map is its own inverse, $P^2=1$. We also have $(PPS)^2=(PP)SPPS=S(PP)S=SS=1$.

Note that this paper makes no claims about the acoustic properties of either the set H of major and minor seventh cords utilized here, nor the transformations and their acoustic effect on the set H. As pointed out in [1], acoustic properties of consonant triads are well established, and the potential of consonant triads to engage in parsimonious voice-leading is due to the group-theoretic properties of their mappings to Z_{12}. Perhaps the PS-group or a similarly constructed algebraic entity can induce similar acoustic properties when applied to certain groups of seventh chords.

Future Work

This paper described the *PS-group* that acts on the set of major and minor seventh chords, which is only one mapping possibility regarding the seventh chords. It would be interesting to investigate whether some other classes of seventh triads, such as diminished, dominant, augmented, or any other of the eight different seventh chord classes, possess properties similar to the ones described here. Furthermore, it might be the case that techniques similar to those described here can be applied to other types of chords containing a number of different pitch classes, in addition to triads and seventh chords. We saw that *T/I-group*, as well as the *PLR-group*, act on specific sets of triads. The T/I-group also acts on the set of major and minor seventh chords, but the results of the group's actions are different from those applied on the set of consonant triads. Lastly, this paper introduced the PS-group, which acts on the set of major and minor seventh chords. Another interesting possibility would be to investigate whether other groups exist, in addition to the ones isomorphic to the dihedral group D_{12} of order 24, that act on various sets of chords of different orders. Such groups have a potential to produce different kinds of harmonic results by acting on specific sets of triads, as well as specific sets of seventh chords.

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