Harmonic Perspective

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Abstract

In projective geometry basic incidence relations and duality are primary. Perspective is an elementary relation between points or lines linked to perspective drawing. Harmonics are little known but pervasive geometric invariants featuring a fundamental fourness divided into two pairs. We discuss several definitions of harmonics, from ones involving basic incidence constructions to non-projective ones with circles, angles and distance ratios. This preliminary study provides a glimpse into the possibilities of harmonic perspective. We describe three artworks that explore harmonics, perspective and their interrelationships. They are composed on hardboard painted with acrylic and oil.

Introduction: A Study of Perspective and Harmonics

This study explores some ideas to develop *harmonic perspective* as a form of artistic representation. The project coalesced when we found a section on harmonic perspective in Hatton's 1913 book [5, pp. 84–5]. In order to sidestep the influences of Renaissance and photographic perspective, Jeannie's pieces create 3D spaces with intersecting planes. CJ's diagrams made with Geogebra [1] help clarify the geometry. First, we present a whirlwind description of the geometry of perspective and harmonics.

Our story begins with the *geometry of duality* which characterizes projective geometry [7, p. 57]. In a 2D plane, duality means that for each geometrical fact there is another formed by replacing "point" with "line" and vice versa. In 3D space, "point" and "plane" swap roles while "lines" are self-dual. The simplest mapping in plane geometry associates points and lines in dual arrangement: the points on a line form a *range*; the lines through a point form a *pencil*. The correspondence between a pencil and a range is a basic *projection*. Next a *perspective* relation joins a pencil with two ranges or a range with two pencils; that is, by combining two elementary projections. A perspectivity maps points to points, or dually, lines to lines as shown in Figures 1 and 2 [3, p. 242][4, pp. 8–13].

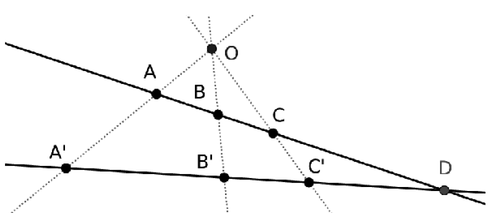

Figure 1 : *Point Perspective. Two lines are perspective from a point if corresponding points on each line lie along lines through the* center *of perspective. We write* $ABCD \overset{O}{\barwedge} A'B'C'D$.

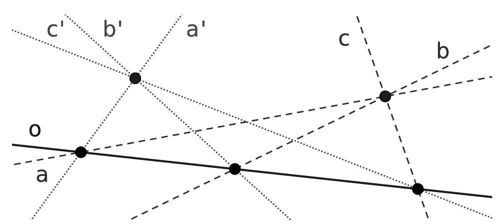

Figure 2 : *Line Perspective. Two points are perspective from a line if corresponding lines in each pencil meet at points along the* axis *of perspective. We write* $abc \overset{o}{\barwedge} a'b'c'$.

The adjective "harmonic" frequently occurs with the words "range," "pencil," "conjugate," "homology," etc. in projective geometry [4, pp. 22-3, 28-9, 30-2, 35, 47-8, 55-6, 73-4]. We use the word "*harmonics*" to synthesize these phenomena into a multi-faceted whole that emphasizes the inherent duality. Below we provide several characterizations to illuminate their nature and pervasiveness. Artistic views are provided by the three painted constructions. With each we provide an adjacent diagram to illustrate some of the geometrical themes underlying the composition. In Figure 4, *Here Kitty*, harmonics are set up using a complete quadrangle and its diagonal points to find the conjugates. Various quadrangles in various planes intersect in a common harmonic range. Different quadrangles use different diagonal and conjugate points.

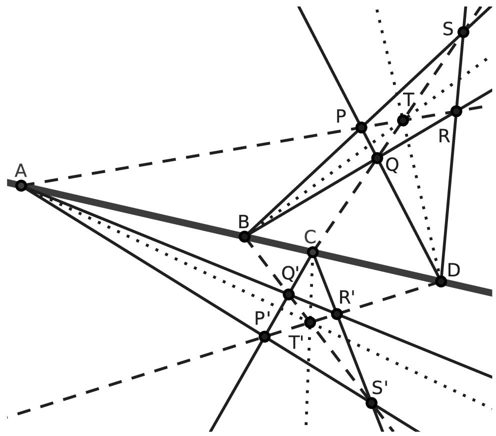

Figure 3: *The Harmonic Property*

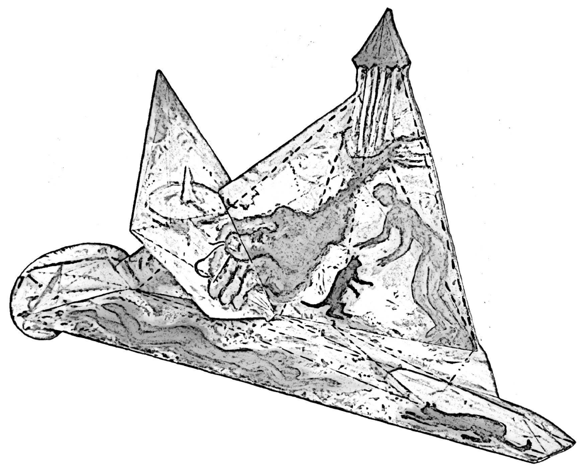

Figure 4: *Here Kitty*

Our first geometric model of harmonics is provided by a *harmonic quadruple* of four points along a line. In Figure 3, given three collinear points B, C and D, their fourth harmonic point A is constructed by choosing a point S anywhere off line $\overleftrightarrow{BCD}$ and a distinct point P on $\overleftrightarrow{BS}$. Draw $\overleftrightarrow{CS}$ and $\overleftrightarrow{DP}$ meeting at Q. Draw $\overleftrightarrow{BQ}$ and $\overleftrightarrow{DS}$ meeting at R. The intersection of $\overleftrightarrow{RP}$ and $\overleftrightarrow{BCD}$ is the unique fourth point A. Notice how the pair B & D are in a special relationship to the *complete quadrangle* $PQRS$: they are two of its three *diagonal points* where its six sides meet in pairs (in a projective plane all lines meet, so there are $\binom{4}{2}$ joins of four points). The *conjugate* pair A & C is cut by the two sides of $PQRS$ that meet at the third vertex T of the quadrangle's *diagonal triangle* TBD. The four points form a *harmonic range* $\mathsf{H}(BD, AC)$. Using quadrangle $P'Q'R'S'$ we find the ordering of the pairs permute, that is, $\mathsf{H}(AC, BD)$. A dual construction for the fourth harmonic line in a pencil using a *complete quadrilateral* is given in [2, p. 19].

Two sides of a quadrangle meet two sides of its diagonal triangle in a point to form a *harmonic pencil*. These four lines are projected from the harmonic range on the side opposite its center. In Figure 3, we say that the lines $\overleftrightarrow{PTR}$ and $\overleftrightarrow{QTS}$ are harmonic conjugates with respect to $\overleftrightarrow{BT}$ and $\overleftrightarrow{DT}$. This construction shows that a quadrangle induces harmonics upon each side and each vertex of its diagonal triangle. More generally, any concurrent set of lines that project from a harmonic range is a harmonic pencil; any collinear set of points that project from a harmonic pencil is a harmonic range. Harmonic ranges and pencils co-occur.

An involution is a transformation of period two, so it is its own inverse. The swapping of conjugate points in an involution of a line on itself with two fixed points provides another definition of harmonics [4, p. 48]. Concretely, in Figure 3, if we designate $M = \overleftrightarrow{BT} \cdot \overleftrightarrow{RS}$ where $\cdot$ means intersection and $N = \overleftrightarrow{BT} \cdot \overleftrightarrow{PQ}$, then $ACBD \overset{T}{\underset{\wedge}{=}} RSMD \overset{B}{\underset{\wedge}{=}} QPND \overset{T}{\underset{\wedge}{=}} CABD$. So the product of the three perspectivities interchanges the conjugate points A and C while leaving the diagonal points invariant. Another involution shows fixed points and conjugates can swap roles (use $P'Q'R'S'$). There is a dual notion for the involutions of a pencil on itself.

For completeness, we note a few more purely projective characterizations of harmonics. In the plane, polarities are point $\Leftrightarrow$ line mappings of period two which induce involutions on whole classes of lines and pencils. When these have two fixed points, they define conics and a proliferation of harmonics. A harmonic range or pencil is specified whenever its cross-ratio as expressed in terms of homogeneous projective coordinates is -1. Finally, the harmonic property is preserved through any nonsingular linear transformation including collineations, correlations, and polarities. Thus harmonics are invariants in geometry.

By introducing distance and angle, we put projective geometry in the background and work in a Euclidean space. Harmonics abound here too. Figure 6, *Inside Out*, features an Appollonian circle construction and an inversion through separate conics so the observer observes the observer iteratively.

At each vertex of a triangle its internal and external angle bisectors divide its legs to form a harmonic pencil. This pencil projects to its opposite side to form a harmonic set. In Figure 5, triangle BDQ with bisectors $\overleftrightarrow{AQ}$ and $\overleftrightarrow{CQ}$ has a harmonic pencil at Q and a harmonic range on its opposite side $\mathsf{H}(BD, AC)$. To see the harmonics from a quadrangle, use $PQRS$ where $\overleftrightarrow{DR}$, $\overleftrightarrow{BP}$ and $\overleftrightarrow{CQ}$ meet at the point S at ∞. A point and its image under inversion in a circle divides the segment between them harmonically. In Figure 5, B is the inverse of D because $\overline{BO} \times \overline{DO} = [\overline{CO}]^2 = [\overline{AO}]^2$ [3, pp. 77–8].

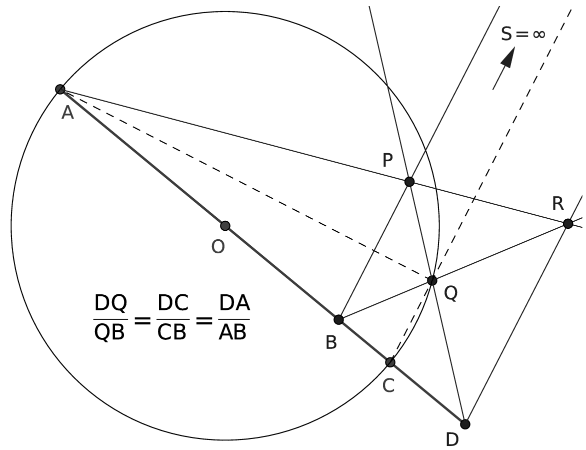

Figure 5 : *The Circle of Appolonius*

Figure 6 : *Inside Out*

The circle of Appollonius is given by the locus of points Q whose distance from two fixed points D and B are in a fixed ratio. We have $\frac{DQ}{QB} = \frac{DC}{CB} = \frac{DA}{AB}$ [3, pp. 88–9]. A geometric expression for this harmonic ratio reads: the outer segment is to the central segment as the whole is to the other outer, which also gives $\frac{AB}{BC} = \frac{AD}{DC}$. For the artist, these *harmonic ratios* provide a means of comparing the sizes of objects in a perspective drawing (the dual form with ratios of angles in a harmonic pencil is also useful). This is explored in the artworks in Figures 4, 6, and 8 where $\frac{\text{person}}{\text{cat}} = \frac{\text{hand}}{\text{needle}}$, $\frac{\text{person}}{\text{window}} = \frac{\text{window}}{\text{cat}}$, $\frac{\text{person}}{\text{scissor}} = \frac{\text{insect}}{\text{hand}}$, respectively.

In addition to these ratios, we note $\overline{DA}$, $\overline{CA}$, and $\overline{BA}$ are in *harmonic progression* where $\overline{CA}$ is the *harmonic mean* between $\overline{DA}$ and $\overline{BA}$. These ratios and progressions provide the link to harmonics in music. If we relabel so that $\overline{ABCD}$ becomes $\overline{OGEC}$, and the stretched string $\overline{OC}$ is tuned to the note C, $\overline{OG} = 10$, $\overline{GE} = 2$ and $\overline{EC} = 3$, then the string held at G and E will complete the major triad CGE [4, pp. 23,135].

Harmonic Perspective

One way to see the emergence of harmonic perspective begins with any two lines and applies the projective axioms that any two lines in a plane meet and that any two points determine a unique line. In Figure 8, *Intentional Cut*, harmonic pencils derived from two lines intersect in a quadrangle whose points will act in the involution of conjugate points along the harmonic line on the left which has one distant invariant point. Given two lines $\overleftrightarrow{EG}$ and $\overleftrightarrow{JH}$ meeting at A with $\overleftrightarrow{EJ}$ and $\overleftrightarrow{GH}$ meeting at C. We have a quadrangle $EGHJ$ with diagonal triangle ACL. So there are harmonic pencils at each of its vertices A, C, and L and harmonic ranges on each of its sides, in particular, $\mathsf{H}(AL, KU)$, $\mathsf{H}(CL, FV)$, and $\mathsf{H}(AC, BD)$ (where $D = \overleftrightarrow{AC} \cdot \overleftrightarrow{LJ}$ is off the page). In this last harmonic set, an involution for swapping A and C is given by this product of three perspectivities $ACBD \overset{T}{\doublebarwedge} PSBU \overset{D}{\doublebarwedge} QRBV \overset{T}{\doublebarwedge} CABD$. Using quadrangle $PQRS$ with diagonal triangle BTD, another reverberation of harmonic ranges and pencils resound.

If we think of this scene dynamically, the points J and H can move along line $\overleftrightarrow{JH}$ which in turn moves around its pivot at A. The invariance of harmonics guarantees that all relationships (quadrangles, involutions, conics, harmonic ratios, etc.) will be preserved throughout the transformation.

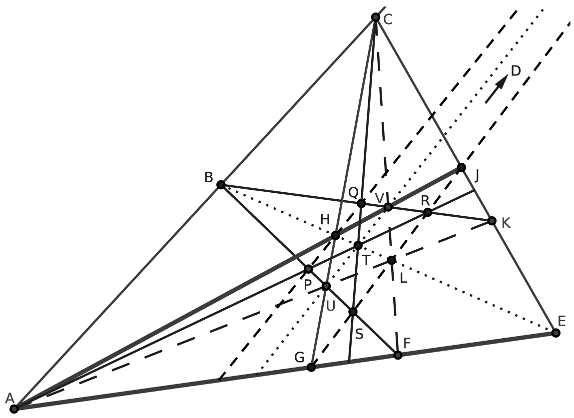

Figure 7 : *Harmonic Perspective*

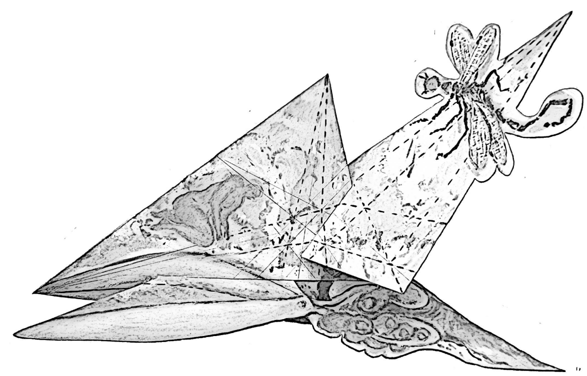

Figure 8 : *Intentional Cut*

This first attempt to develop art from harmonic perspective was very illuminating, but we think a more satisfying integration of art and projective geometry is possible. Is it possible to design art based on a geometrical invariant such as harmonics? Can harmonic perspective help us understand the simultaneous perceptions of large and small, near and far objects? What qualities can harmonic perspective interject into a work of art? The many definitions of harmonics provide tools for the artist and the geometer to explore these and other representational questions. It is a fruitful area to study because the ubiquity of these forms and their interrelationships suggest that projective geometry informs fundamentals of spatial experience.

References

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